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TEACHIING ADAPTIVE AND STRATEGIC REASONING THROUGH FORMULA DERIVATION: BEYOND FORMAL SEMIOTICS

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The paper illustrates how mathematical tasks involving dynamic solution methods such as formula derivations can strengthen mathematical proficiency for secondary level students through exposure to *adaptive* and *strategic* reasoning.

Keywords: Adaptive Reasoning; Strategic Competence; Procedural Fluency; Solution Dynamics; formula derivation

1. Introduction

Conventional logic in teaching American secondary level students (*secondary level* meaning students comprised of grades 7-12) tells us that mathematics instruction should be both efficient and informative. Logic also suggests that efficiency and information should not be mutually exclusive, and yet American students continue to struggle with adapting their learning beyond the rudiments of their textbooks. This is possibly because the most common instructional method used in secondary mathematics classrooms is highly structured and is based primarily on the semiotic processes so commonly illustrated in mathematics textbooks (Watanabe, 2007). *Semiotic*s in mathematics is simply defined as the use of symbols that are helpful in understanding the processes of thinking, symbolizing, and communicating (Radford, Schubring, & Seeger, 2008). Semiotics in mathematics instruction typically refers to the teaching of mathematics as a process of symbol manipulation through structured algorithms and rigorously defined theorems. The outcome for students in this strategy of teaching and learning is referred to as *procedural fluency*. Procedural fluency is defined as the skill in carrying out procedures flexibly, accurately, efficiently, and appropriately (National Research Council, 2001). Procedurally fluent students ostensibly develop the ability to evaluate and simplify various expressions, solve simple equalities, and represent mathematical relationships in graphical form. However, as the mathematics performance of American secondary students continues to remain low relative to students in other countries, the value of teaching for procedural fluency without identifying the relevance of the underlying mathematical logic remains a matter of debate. The manuscript at hand describes how pedagogy including *Strategic Competence* and *Adaptive Reasoning* can supplement procedural fluency to create a more comprehensive learning experience for secondary mathematics students.

Adaptive Reasoning and Strategic Competence have emerged as two critical components of mathematical proficiency in contemporary teacher training programs in the United

States. *Adaptive Reasoning* is loosely defined as the capacity for logical thinking and the ability to reason and justify why solutions are appropriate within the context of problems that are large in scope, while *Strategic Competence* refers to the ability to formulate suitable mathematical models and select efficient methods for solving problems (National Research Council, 2001). These mathematical strands, along with procedural fluency, are not intended to be independent nor are they intended to be instructed as isolated skills. They are actually highly interdependent and represent different aspects of a more comprehensive kind of mathematics teaching and learning. The examples presented hereafter illustrate how mathematical representations such as formula derivations can be used to develop a balance of procedural fluency, adaptive reasoning, and strategic competence.

The ability to incorporate adaptive reasoning and strategic competence into mathematics instruction is challenging to be certain. Research on teacher lesson planning suggests that reform-based curricula in mathematics presents contemporary challenges that are much different than those lesson development challenges seen within more conventional or traditional curriculums (Superfine, 2008). Thus, the development of instructional tasks that are dynamic in how they present adaptive or strategic tasks is difficult. Nevertheless, adaptive reasoning and strategic competence remain critical elements of a coherent understanding of mathematics. Romberg (2000) argues that with appropriate guidance from teachers, students can build a coherent understanding of mathematics, and that their understanding about how the symbolic processes of mathematics can evolve into increasingly abstract and scientific reasoning. This suggests that supplementing traditional procedural fluency problems with intentional efforts to include adaptive and strategic undertones results in higher levels of comprehension.

The long-term benefits of mathematical exercises, such as those that follow, are also a matter of debate. Many American teachers consider mathematical tasks requiring students to exhibit *adaptive* and *strategic* processing to be vague and inadequate for developing specific (and testable) mathematical skill sets. Mathematical tasks, such as the formula derivations presented herein, have several possible solutions and do not typically emphasize the development any specific kind of procedural fluency. For this reason, they are often viewed as inefficient or even superfluous exercises in the classroom. The counterpoint, however, suggests that vaguely defined tasks are exactly what allow *strategic* and *adaptive* reasoning to occur in a mathematics lesson. A mathematical task having the potential to be completed using numeric, algebraic, geometric, or even calculus-based approaches is especially powerful. The National Council of Teachers of Mathematics (NCTM, 2000) suggests that teachers should use a variety of strategies that allow them to monitor students' capacity to analyze mathematical situations, frame and solve problems and better make sense of the procedures they have been taught. Each of these recommended methods and strategies provides insights into *strategic* and *adaptive* reasoning.

Because the solutions to broader scope mathematical tasks are often dynamic and depend on the mathematical preparedness of the student, the identical mathematical exercises can be most effectively used successively over a period of years to further emphasize the nature and necessity of strategic thinking (Ostler, Grandgenett, & Mitchell, 2008). Students would select and develop appropriate mathematical models and efficient methods of solving problems based on what their level of mathematical competence allows. These kinds of problems are known as Dynamic Solution Exercises (DSE) and accentuate the need to learn increasingly sophisticated mathematical strategies as a way to more efficiently complete mathematical tasks.

2. Algebraic Formula Derivation (DSE Level 1)

Suppose we want students to present an algebraic solution for deriving the formula for calculating the volume of the frustum of a cone. There are many ways to complete this task. Some are simple while others are more sophisticated and mathematically rigorous. For example, we may begin by constructing a cone for which we already know the formula and cut it to create a frustum. A geometric strategy would be to construct a model that represents the large cone as a frustum and a smaller cone resting on the top (**Fig. 1**). This is a *strategic* model and demands a justification that is more sophisticated than what student might see with traditional semiotic processing. Questioning how and why this method works is what provides the *adaptive* and *strategic* learning.

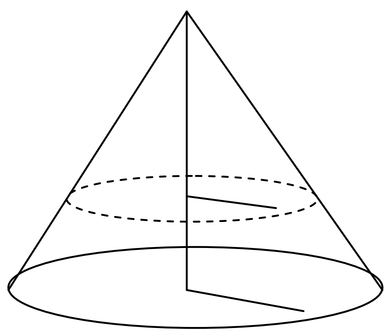

Fig. 1: Cone and frustum

We will assume, base on previous experience, that we know the formula for the volume of a cone. Given this assumption, we then develop algebraic notation for the total volume of the large cone as the sum of the small cone and the frustum. The frustum then must be the difference of the large cone and the small cone. Students would need to justify the algebraic representations of the geometric model as follows:

$$V_{cone} = \frac{\pi}{3} r^2 h \qquad (1)$$

$$V_t = V_f + V_c \qquad (2)$$

$$V_f = V_t - V_c \qquad (3)$$

Defining the variables represents an *adaptive* task because students are not often asked to assign variables in equations they must derive. For the most part, students are simply required to manipulate existing symbols. For this derivation we will label relevant variables as follows: ***k*** (height of the small cone), ***r*** (radius of the base of the small cone), ***h*** (height of the frustum), and ***R*** (radius of the base of the frustum and of the large cone). Substituting the newly defined variables into the equation for a cone (1) and substituting each respective volume equation into equation (3) gives us the following series of equations. Once again the strategic reasoning would need to be emphasized in the classroom.

$$V_f = V_t - V_c \tag{4}$$

$$V_f = \frac{\pi}{3} R^2 (h + k) - \frac{\pi}{3} r^2 k \tag{5}$$

$$V_f = \frac{\pi}{3} R^2 h + \frac{\pi}{3} R^2 k - \frac{\pi}{3} r^2 k \tag{6}$$

We now have in Eq. (6) a general formula that can be simplified to complete our derivation; however, ***k*** does not formally exist in the frustum, so students must draw from previous algebraic experiences to define ***k*** in terms of the existing variables. The following method illustrates the use of a *proportion*, which requires both adaptive reasoning and strategic competence. Although students may be procedurally proficient at completing proportion tasks, they often experience great difficulty in this kind of logic because the opportunities to adapt algebraic procedures from one task in order to make them appropriate for another are very limited. We solve the proportion for ***k*** in terms of our existing variables and substitute the result shown in Eq. (8) into our frustum Eq. (9). The remainder of the derivation is primarily algebraic; however, note that in Eq. (10) a *rational* form of "one" is necessary to simplify the equation. Students often miss this step.

$$\frac{k}{r} = \frac{h + k}{R} \tag{7}$$

$$k = \frac{hr}{R - r} \tag{8}$$

$$V_f = \frac{\pi}{3} \left[R^2 h + R^2 \left(\frac{hr}{R - r} \right) - r^2 \left(\frac{hr}{R - r} \right) \right] \tag{9}$$

$$V_f = \frac{\pi}{3}\left[\frac{R^2h(R-r)}{R-r} + \left(\frac{R^2hr}{R-r}\right) - \left(\frac{r^2hr}{R-r}\right)\right] \qquad (10)$$

$$V_f = \frac{\pi}{3}\left[\frac{R^3h}{R-r} - \frac{R^2rh}{R-r} + \frac{R^2hr}{R-r} - \frac{r^2hr}{R-r}\right] \qquad (11)$$

$$V_f = \frac{\pi h}{3}\left[\frac{R^3}{R-r} - \frac{R^2r}{R-r} + \frac{R^2r}{R-r} - \frac{r^3}{R-r}\right] \qquad (12)$$

$$V_f = \frac{\pi h}{3}\left[\frac{R^3}{R-r} - \frac{r^3}{R-r}\right] \qquad (13)$$

In Eq. (14) below, students are often tempted to consider the task complete. Once again, a simple but necessary factoring procedure requires adaptive reasoning because the need to factor is not immediately apparent. The factoring of the *difference of cubes*, though taught and practiced, is not commonly contextualized in broader problems such as this derivation.

$$V_f = \frac{\pi h}{3}\left[\frac{R^3 - r^3}{R-r}\right] \qquad (14)$$

$$V_f = \frac{\pi h}{3}\left[\frac{(R-r)(R^2 + Rr + r^2)}{R-r}\right] \qquad (15)$$

$$V_f = \frac{\pi h}{3}\left(R^2 + Rr + r^2\right) \qquad (16)$$

The student outcome of this task is exposure to a simple algebraic derivation that makes complex assumptions about the kinds of adaptive reasoning abilities that students possess. The point of the example is that procedural fluency in *solving proportions*, *renaming fractions*, or *factoring polynomials* does not necessarily mean students are able to adapt the procedures for use in tasks such as the previous formula derivation. Special instructional emphases on how these adaptive and strategic processes are carried out is necessary in order for students to become proficient in these kinds of skills. Additionally, it is worthwhile to compare the value of the algebraic method used above with a more sophisticated procedure in order to develop advanced *strategic* skills. Students must be exposed to *strategic* thinking to learn to model tasks such as the frustum so that appropriate methods can be selected and carried out in similar tasks. Strategic thinking is often the result of witnessing a kind of problem, such as a formula derivation, completed using a number of different strategies and then demonstrating the ability to select and use the most efficient method for similar kinds of tasks. This is a form of *Solution Dynamics* (i.e. the solution process is a *dynamic* process depending on the preparation and ability of the student) that allows for the completion of a single problem at multiple levels. An illustration of this would be to complete the frustum formula derivation in a more sophisticated way. One possible calculus-based solution for deriving the formula for the volume of a frustum is illustrated below.

3. Calculus Formula Derivation (DSE Level 2)

In this derivation, the process of defining the model and the variables of radii and height for the frustum is more formal than what was done using the algebraic method. We use ordered pairs to define distances rather than assigning variable lengths to arbitrary symbols as was done in the algebraic example. This makes the variables more tangible to students, but also makes adapting certain algebraic processes more difficult for them. Instruction in how variables are assigned is important and must include opportunities to explore the corroborating algebraic relationships since the lengths of different parts of the frustum are derived from coordinate values within the ordered pairs, and the shape of the frustum itself is defined by the equation of a general linear equation.

The process used in the following example is calculus-based and is difficult for students from a strategic standpoint only in that they must recognize how to translate a single-integration area model to compute volume by using a disk-method. This is done using a general linear equation and rotating it around an axis. The remainder of the problem is procedural and largely algebraic, but it is still worth comparing to the previous method for strategic purposes. In particular, the strategy does not use the differences of two cones to geometrically define the frustum. Instead, a simple linear equation is used to define the frustum. In this example the frustum starts with center point of the top circular surface at the point (0,0) and with the bottom circular surface terminating at the point (***h***,0). A general linear equation using points at the end of each respective radius, the top defined by the point (0,***r***) and the bottom defined by the point (***h***,***R***), determine the slant of the frustum as illustrated in **Fig. 2**.

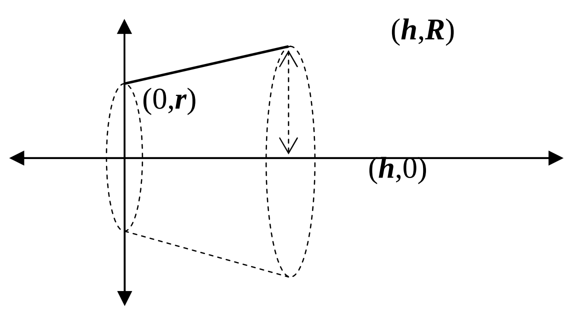

Fig. 2: Frustum with top and bottom bases at (0,0) and (h,0) respectively; and small and large radii of r and R respectively.

Note that the equation of the line defined by the given points must be derived using slope-*y* intercept equation. The slope is then used to define general linear equation, which can be rotated around the x-axis to create a series of disks. This is both an adaptive and a strategic skill. It is helpful to have students justify the parts of the resulting differential equation and describe the process of adding a series of disks whose radii are successively determined by the height of the liner function. The purpose is to demonstrate the adaptive reasoning needed for describing how discrete calculations are transferred to continuous methods.

$$m = \frac{R-r}{h-0} = \frac{R-r}{h} \tag{17}$$

$$f(x) = \left(\frac{R-r}{h}\right)x + r \tag{18}$$

Eq. (18), the function that will represent the radius of each of the disks, is going to be substituted in for the radius variable (***r***) in Eq. (19).

$$\int_0^h \pi r^2 dx \tag{19}$$

$$\int_0^h \pi \left[f(x)\right]^2 dx \tag{20}$$

$$\int_0^h \pi \left[\left(\frac{R-r}{h}\right)x + r\right]^2 dx \tag{21}$$

$$\pi \int_0^h \left[\left(\frac{R-r}{h}\right)^2 x^2 + 2r\left(\frac{R-r}{h}\right)x + r^2\right] dx \tag{22}$$

A few basic rules of integration are applied and the antiderivative calculated before the variable ***h*** is substituted into Eq. (24). The resulting algebra yields Eq. (29), which is the same volume formula for the frustum as was derived previously.

$$\pi\left[\left(\frac{R-r}{h}\right)^2\frac{x^3}{3}+2r\left(\frac{R-r}{h}\right)\frac{x^2}{2}+r^2x\right]_0^h \tag{23}$$

$$\pi\left[\left(\frac{R^2-2Rr+r^2}{h^2}\right)\frac{h^3}{3}+\left(\frac{2Rr-2r^2}{h}\right)\frac{h^2}{2}+r^2h\right]-\pi[0] \tag{24}$$

$$\pi\left[\left(\frac{R^2-2Rr+r^2}{1}\right)\frac{h}{3}+\left(\frac{Rr-r^2}{1}\right)\frac{h}{1}+r^2h\right] \tag{25}$$

$$\pi\left[\left(\frac{R^2-2Rr+r^2}{1}\right)\frac{h}{3}+\left(\frac{Rr-r^2}{1}\right)\frac{(3)h}{(3)1}+\frac{(3)r^2h}{(3)}\right] \tag{26}$$

$$\frac{\pi h}{3}\left[R^2-2Rr+r^2+\left(\frac{Rr-r^2}{1}\right)\frac{(3)}{1}+\frac{(3)r^2}{1}\right] \tag{27}$$

$$\frac{\pi h}{3}\left[R^2-2Rr+r^2+3Rr-3r^2+3r^2\right] \tag{28}$$

$$V_f=\frac{\pi h}{3}\left(R^2+Rr+r^2\right) \tag{29}$$

Alternative methods for completing this task (using calculus) may initially seem redundant but they are important for the development of both procedural fluency and strategic competence. Students who are not given the opportunity to see how dynamic processes develop are not able to adapt their thinking to other, more complex mathematical situations.

4. A Comparative Example of Strategic Competence

In the introduction of this manuscript, strategic competence was roughly defined as the ability to formulate suitable mathematical models and select efficient methods for solving problems. Consider the following derivation. Suppose we want to derive the formula for calculating the area of a circle. Logic, of course, suggests that we simply use a known reference; however, if the task is to demonstrate the ability to derive the equation, then the *strategic* task is to select the most efficient task using available information. There are many forms of algebraic, geometric, trigonometric, and calculus based derivations for completing the circle area formula, but extending our strategic abilities for the purposes of efficiency might look like the following comparison of two methods. In the first method, the circle is graphed on a coordinate plane with the center at (0,0) and with a radius of ***r***. We then solve for *y*, representing it as a function and integrate the resulting form from 0 to ***r***. The resulting formula represents only one fourth of the circle. The derivation is illustrated below:

$$x^2 + y^2 = r^2 \tag{30}$$

$$y = \sqrt{r^2 - x^2} \tag{31}$$

$$\int_0^r \sqrt{r^2 - x^2}\ dx \tag{32}$$

$$\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a} \tag{33}$$

$$\frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2}\sin^{-1}\frac{x}{r}\Big|_0^r \tag{34}$$

$$\left[\frac{r}{2}\sqrt{r^2 - r^2} + \frac{r^2}{2}\sin^{-1}\frac{r}{r}\right] - \left[\frac{0}{2}\sqrt{r^2 - 0^2} + \frac{r^2}{2}\sin^{-1}\frac{0}{r}\right] \tag{35}$$

$$\left[\frac{r}{2}(0) + \frac{r^2}{2}\sin^{-1}1\right] - \left[\frac{0}{2}(r) + \frac{r^2}{2}\sin^{-1}0\right] \tag{36}$$

$$\left[0 + \frac{r^2}{2}\cdot\frac{\pi}{2}\right] - \left[0 + \frac{r^2}{2}(0)\right] \tag{37}$$

$$\frac{\pi r^2}{4} \quad (38)$$

Now let us consider a different more simple and efficient technique using the known circumference formula of a circle. Visualize cutting roughly circular strips from the circle that look like thin circular bands. If the bands are cut, they become roughly rectangular. The lengths of the rectangular strips are analogous to the circumference of the circle and the widths of the strips are the differences in ***r*** from the inside of each strip to the outside. Adding all of the strips together by integrating from the center of the circle to the edge of the circle at radius ***r*** will result in the area formula we seek. Well conceived, this is simply a matter of integrating the circumference over the length of the radius.

$$\int_0^r 2\pi r \; dr \quad (39)$$

$$2\pi \int_0^r r \; dr \quad (40)$$

$$2\pi \left(\frac{r^2}{2} \right) \Big|_0^r \quad (41)$$

$$\pi r^2 \quad (42)$$

The derivation is completed in a few simple steps, which represents strategic insight. This method can also be easily adapted to a three dimensional model where the surface area of a sphere can be integrated to derive the formula for the volume. Repeated exposure to such derivations not only provides procedural fluency for different maths, it also lends credibility to strategic and adaptive reasoning for fundamentally equivalent mathematical tasks.

5. Conclusion

The focus on using formulas has become so common in secondary mathematics instruction that we have underestimated the value of contextualizing the processes that allows them to exist. Perhaps the derivations presented previously illustrate an over simplified approach to a very complex problem, but unless teachers truly consider the innate limitations of the way we teach procedural fluency, we will perpetuate the belief that procedures alone represent what constitutes good mathematical learning and what makes good mathematicians.

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