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CHORD LENGTH DISTRIBUTION FUNCTION FOR CONVEX POLYGONS

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Using the inclusion-exclusion principle and the Pleijel identity, an algorithm for calculation chord length distribution function for a bounded convex polygon is obtained. In the particular case an expression for the chord length distribution function for a rhombus is obtained.

Keywords: chord length distribution function; Pleijel identity; rhombus.

1. INTRODUCTION

Let G be the space of lines g in the Euclidean plane R^2, (p, φ) = the polar coordinates of the foot of the perpendicular to g from the origin O, be standard coordinates for a line $g \in \mathrm{G}$.

Let $\mu(\cdot)$ stand for locally finite measure on G invariant with respect to the group of all Euclidean motions (translations and rotations). It is well known that the element of the measure up to a constant factor has the following form (see [1], [2]):

$$\mu(dg) = dg = dp\, d\varphi,$$

where dp is one dimensional Lebesgue measure, while $d\varphi$ is the uniform measure on the unit circle.

For each bounded convex domain D the set of lines that intersect D we denote by

$$[\mathrm{D}] = \{g \in \mathrm{G} : \ g \cap D \neq \emptyset\}$$

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and we have (see [1], [2]):

$$\mu([\mathbb{D}]) = |\partial\mathbb{D}|,$$

where $\partial\mathbb{D}$ is the boundary of $\mathbb{D}$ and $|\partial\mathbb{D}|$ stands for the length of $\partial\mathbb{D}$.
Let $A^y_{\mathbb{D}}$ be the set of lines that intersect $\mathbb{D}$ producing a chord $\chi(g) = g \bigcap \mathbb{D}$ of length less than or equal to y:

$$A^y_{\mathbb{D}} = \{g \in [\mathbb{D}] : \ |\chi(g)| \le y\}, \qquad y \in \mathbb{R}.$$

Distribution function of the length of a random chord χ of $\mathbb{D}$ is defined as

$$F(y) = \frac{1}{|\partial\mathbb{D}|}\,\mu(A^y_{\mathbb{D}}) = \frac{1}{|\partial\mathbb{D}|}\iint_{A^y_{\mathbb{D}}} d\varphi\, dp. \tag{1.1}$$

Therefore, to obtain chord length distribution function for a bounded convex domain $\mathbb{D}$ we have to calculate the integral in the right-hand side of (1.1). Explicit formulae for the chord length distribution functions are known only for the cases of a disc, a rectangle [3], and a regular polygon [5].
The main result of the paper is an algorithm for calculation the chord length distribution function for a bounded convex polygon. In particular, an expression for the chord length distribution function for a rhombus is obtained.
The determination of the chord length distribution function has a long tradition of application to collections of bounded convex bodies forming structures in metal and ceramics. The series of formulae for chord length distribution functions may be of use in finding suitable models when empirical distribution functions are given (see [5], [6], [9] and [12]).
In [11] Sulanke proved that $\chi(g) = g \cap D$ is a measurable function on $[D]$. He also proved that the function $F(y)$ defined by formula (1.1) is a continuous function with respect to y. In [8] Gates obtained some properties and inequalities for the chord length distribution function for any bounded convex domain on the plane. In particular, he proved that in the case of a bounded convex polygon the chord length distribution function is expressed in terms of elementary functions.

2. THE CASE OF A CONVEX POLYGON

Let $\mathbb{D}$ be a convex bounded polygon in the plane and a_1, a_2,..., a_n be sides of $\mathbb{D}$. Then

$$[\mathbb{D}] = \bigcup_{i<j}([a_i] \cap [a_j])$$

where $[a_i] \cap [a_j]$ is the set of lines hitting both sides a_i and a_j of $\mathbb{D}$.
We write

$$F(y) = \frac{1}{|\partial\mathbb{D}|}\sum_{i<j}\iint_{\{g\in[a_i]\cap[a_j]:|\chi(g)|\le y\}} d\varphi\, dp =$$

$$= \frac{1}{|\partial\mathbb{D}|}\left[\sum_{i<j}^{I}\iint_{\{g\in[a_i]\cap[a_j]:|\chi(g)|\le y\}} d\varphi\, dp + \sum_{i<j}^{II}\iint_{\{g\in[a_i]\cap[a_j]:|\chi(g)|\le y\}} d\varphi\, dp\right],$$

where $\sum^{I}$ is over all pairs of nonparallel segments a_i, $a_j \subset \partial\mathbb{D}$ and $\sum^{II}$ is over all pairs of parallel segments a_i and $a_j \subset \partial\mathbb{D}$.

In each of the integrals in the sum $\sum\limits_{i<j}^{I}$ one integration can be performed by passing to the $(|\chi|, \varphi)$ coordinates. We have (see [2], page 157):

$$dg = \frac{\sin\alpha_1\, \sin\alpha_2}{|\sin(\alpha_1+\alpha_2)|}\, d|\chi|\, d\varphi, \tag{2.1}$$

where α_1 is the angle between a_i and $\chi(g) = g \cap \mathbb{D}$, while α_2 is the angle between a_j and $\chi(g)$ $(g \in [a_i] \cap [a_j])$, α_1 and α_2 lie in one half-plane with respect to inside of $\mathbb{D}$.

3. PLEIJEL IDENTITY

Let $\mathbb{D}$ be a convex bounded polygon in the plane and a_1,..., a_n be sides of $\mathbb{D}$. The so–called Pleijel identity for $\mathbb{D}$ is as follows (see [2], page 156):

$$\int_{[\mathbb{D}]} f(|\chi(g)|)\, dg = \int_{\mathbb{G}} f'(|\chi|)\cdot|\chi|\, \cot\alpha_1\, \cot\alpha_2\, dg + \sum_{i=1}^{n}\int_0^{|a_i|} f(u)\, du, \tag{3.1}$$

where $f(x)$ is a function with continuous first derivative $f'(x)$, α_1 and α_2 are the angles between $\partial\mathbb{D}$ and g at the endpoints of $\chi(g) = g\cap\mathbb{D}$ which lie in one half-plane with respect to the inside of $\mathbb{D}$, $|a_i|$ is the length of a_i, $i = 1, ..., n$.

It was shown by R. V. Ambartzumian in [2] (page 156) that the identity (3.1) is useful to calculate chord length distribution function for the case of a bounded convex polygon. If in (3.1) we formally put

$$f_y(u) = \begin{cases} 0 & \text{if} \quad u \le y \\ 1 & \text{if} \quad u > y \end{cases}$$

then the left-hand integral in (3.1) will equal

$$\mu\{g \in [\mathbb{D}] :\ |\chi(g)| > y\},$$

i.e. the invariant measure of the set of chords of $\mathbb{D}$ whose length exceeds y. The derivative of $f_y(u)$ should be replaced by Dirac's δ–function concentrated at y (see [2], page 156). Therefore, we obtain

$$[1 - F(y)]\,|\partial\mathbb{D}| = \sum_{i<j}^{I}\iint_{[a_i]\cap[a_j]} \delta(|\chi| - y)\cdot|\chi|\, \cot\alpha_1\, \cot\alpha_2\, dg +$$

$$+\sum_{i<j}^{II}\iint_{[a_i]\cap[a_j]} \delta(|\chi| - y)\cdot|\chi|\, \cot\alpha_1\, \cot\alpha_2\, dg + \sum_{i=1}^{n}(|a_i| - y)^{+}, \tag{3.2}$$

where $x^{+} = x$ if $x > 0$, and 0 otherwise.

For any continuous function f we have (see [10]):

$$\int_{\mathbb{R}^n} \delta(x-y)\, f(x)\, dx = f(y). \tag{3.3}$$

For each pair $\{a_i, a_j\} \in \sum_{i<j}^{I}$ we make the change of variables $(p, \varphi) \to (|\chi|, \varphi)$ and using (2.1) and (3.3) we obtain

$$\iint_{[a_i]\cap[a_j]} \delta(|\chi|-y)\cdot|\chi| \cot\alpha_1 \cot\alpha_2\, dg = \frac{y}{\sin\gamma_{ij}} \int_{\Phi_{ij}(y)} \sin\varphi\, \sin(\gamma_{ij}-\varphi)\, d\varphi,$$

where γ_{ij} is the angle between nonparallel sides a_i and a_j (or their continuations), φ is the angle between direction φ and direction of segment a_i $i<j$, and

$$\Phi_{ij}(y) = \{\varphi : \text{ a chord joining}\, a_i \,\text{and}\, a_j \,\text{exists with direction } \varphi \text{ and length}\, y\}.$$

Note, that $\Phi_{ij}(y)$ is a subset in the space of directions in the plane, while in the integral in the right hand side the corresponding $\Phi_{ij}(y)$ is the set of angles, where reference direction coincides with direction of segment a_i, $i<j$ and reference origin coincides with the intersection of the lines containing a_i and a_j.

Moreover, for parallel sides a_i and a_j (i.e. a_i, $a_j \in \sum_{i<j}^{II}$) we have

$$\sum_{i<j}^{II} \iint_{[a_i]\cap[a_j]} \delta(|\chi|-y)\cdot|\chi| \cot\alpha_1 \cot\alpha_2\, dg =$$

$$= -\sum_{i<j}^{II} I_{ij}(y) \int \delta(|\chi|-y)\cdot|\chi|\, h(\varphi_\chi) \tan^2\varphi_\chi \frac{d\varphi_\chi}{d|\chi|}\, d|\chi| =$$

$$= -\sum_{i<j}^{II} I_{ij}(y)\, h(\varphi_y) \tan^2\varphi_y \frac{b_{ij}}{\sqrt{y^2-b_{ij}^2}},$$

where φ_y takes on two values $\arccos\frac{b_{ij}}{y}$ or $2\,\pi - \arccos\frac{b_{ij}}{y}$, b_{ij} is the distance between the parallel segments a_i and a_j (i.e. the distance between the lines containing a_i and a_j), and $I(A)$ is the indicator function of the event A, i.e. $I(A)=1$ if A has occurred and 0 otherwise, indicator $I_{ij}(y) = I(\text{length of shortest chord hitting}\, a_i \text{ and } a_j \le y \le \text{length of longest chord hitting } a_i \text{ and}\, a_j)$. Further, $h(\varphi) \neq b_{ij}$ is the height of the maximal parallelogram with two sides equal to $\chi(\varphi) = g(\varphi)\cap\mathbb{D}$, $g(\varphi) \in [a_i]\cap[a_j]$ ($g(\varphi)$ is a line with φ-direction), and the other two sides lie on the parallel sides a_i and a_j,

$$h(\varphi_\chi) = h\left(\arccos\frac{b_{ij}}{|\chi(\varphi)|}\right) + h\left(2\,\pi - \arccos\frac{b_{ij}}{|\chi(\varphi)|}\right).$$

Hence, $h(\cdot)=0$ if the parallelogram is empty.
For the value of φ such that $|\chi(\varphi)| = y$ we have $h(\varphi_y) = h(\varphi_\chi)$.

Therefore we obtain

$$F(y) = 1 - \frac{1}{\sum_{i=1}^{n} |a_i|} \left[\sum_{i<j}^{I} \frac{y}{\sin\gamma_{ij}} \int_{\Phi_{ij}(y)} \sin\varphi \, \sin(\gamma_{ij} - \varphi) \, d\varphi - \right.$$

$$\left. - \sum_{i<j}^{II} I_{ij}(y) \, h(\varphi_y) \, \frac{\sqrt{y^2 - b_{ij}^2}}{b_{ij}} + \sum_{i=1}^{n} (|a_i| - y)^+ \right]. \qquad (3.4)$$

In the case where $\partial\mathbb{D}$ contains no pairs of parallel sides, formula (3.4) coincides with the expression given by R. V. Ambartzumian in [2], page 158.

It follows from (3.4) that to find distribution function $F(y)$ we have to calculate integrals of the form

$$\frac{1}{\sin\gamma} \int_{\Phi_{a,b}(y)} \sin\varphi \, \sin(\gamma - \varphi) \, d\varphi$$

for any two nonparallel segments a and b ($b \le a$) with the angle γ between a and b (or their continuations) and also calculate the second sum in (3.4) (for pairs of parallel sides). Here and below $\Phi_{a,b}(y)$ is

$$\Phi_{a,b}(y) = \{\varphi : \text{ a chord joining } a \text{ and } b \text{ exists with direction } \varphi \text{ and length } y\}.$$

4. THE DOMAIN $\Phi_{a,b}(y)$

Consider segments a and b ($b \le a$), which have one common endpoint and make an angle γ.

The line g with parameters (φ, p) intersect sides a and b, if

$$\varphi \in \left(-\frac{\pi}{2} + \gamma, \frac{\pi}{2} \right).$$

Without loss of generality we can assume, that the reference direction (X-axis) coincides with the direction of the segment a and the origin O coincides with the intersection of the lines containing a and b. Find the intersection point of the line $g = (\varphi, p)$ with the segment a. We have

$$x = \frac{p}{\cos\varphi} \quad \text{and} \quad y = 0.$$

Therefore,

$$g \cap a = \begin{cases} x = \frac{p}{\cos\varphi} \\ y = 0 \\ 0 \le p \le a \, \cos\varphi. \end{cases}$$

Find the intersection point of the line $g = (\varphi, p)$ with the segment b. We have

$$\begin{cases} x \, \cos\varphi + y \sin\varphi = p \\ y = \tan\gamma \, x \end{cases} \quad \text{or} \quad \begin{cases} x = \frac{p \, \cos\gamma_k}{\cos(\varphi - \gamma)} \\ y = \frac{p \, \sin\gamma}{\cos(\varphi - \gamma)} \end{cases}.$$

Hence, we obtain

$$g \cap b = \begin{cases} x = \frac{p\ \cos\gamma}{\cos(\varphi-\gamma)} \\ y = \frac{p\ \sin\gamma}{\cos(\varphi-\gamma)} \\ 0 \le p \le b\cos(\varphi-\gamma). \end{cases}$$

The line g which intersects sides a and b forms a chord χ of the length

$$|\chi| = \frac{p\ \sin\gamma}{\cos\varphi\ \cos(\varphi-\gamma)}.$$

Therefore, we come to the following system of relations:

$$\begin{cases} 0 \le p \le a\cos\varphi \\ 0 \le p \le b\cos(\varphi-\gamma) \\ |\chi| = \frac{p\ \sin\gamma}{\cos\varphi\ \cos(\varphi-\gamma)}. \end{cases} \tag{4.1}$$

If $a\cos\varphi \le b\cos(\varphi-\gamma)$, then $\tan\varphi \ge \frac{a-b\cos\gamma}{b\sin\gamma}$. Hence, $\varphi \ge \arctan\frac{a-b\cos\gamma}{b\sin\gamma}$. It is not difficult to verify, that $\arctan\frac{a-b\cos\gamma}{b\sin\gamma} \in \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right)$. Since $\varphi \in \left(-\frac{\pi}{2}+\gamma, \arctan\frac{a-b\cos\gamma}{b\sin\gamma}\right)$, then (4.1) has the following form:

$$\begin{cases} 0 \le p \le b\cos(\varphi-\gamma) \\ |\chi| = \frac{p\ \sin\gamma}{\cos\varphi\ \cos(\varphi-\gamma)}. \end{cases}$$

Moreover, we get

$$\chi_{\min}(\varphi) = 0, \qquad \chi_{\max}(\varphi) = \frac{b\sin\gamma}{\cos\varphi}.$$

The function $\chi_{\max}(\varphi)$ in the domain $\left(-\frac{\pi}{2}+\gamma, 0\right)$ decreasing, while in the domain $\left(0, \arctan\frac{a-b\cos\gamma}{b\sin\gamma}\right)$ increasing. Hence, we have

$$\varphi_{\min} = 0, \quad \chi_{\max}\left(-\frac{\pi}{2}+\gamma\right) = b, \quad \chi_{\max}(\varphi_{\min}) = b\sin\gamma \quad \text{and}$$

$$\chi_{\max}\left(\arctan\frac{a-b\cos\gamma}{b\sin\gamma}\right) = \sqrt{a^2+b^2-2ab\cos\gamma} = c,$$

where c is the third side of the triangle made by a and b.
Now let $\varphi \in (\arctan\frac{a-b\cos\gamma}{b\sin\gamma}, \frac{\pi}{2})$. In this case (4.1) can be rewritten in the form:

$$\begin{cases} 0 \le p \le a\cos(\varphi)) \\ |\chi| = \frac{p\ \sin\gamma}{\cos\varphi\ \cos(\varphi-\gamma)}. \end{cases}$$

Moreover, we get

$$\chi_{\min}(\varphi) = 0, \qquad \chi_{\max}(\varphi) = \frac{a\sin\gamma}{\cos(\varphi-\gamma)}.$$

If $\arctan\frac{a-b\cos\gamma}{b\sin\gamma} > \gamma$ $\left(\text{i.e. } \cos\gamma > \frac{b}{a} \text{ or } \gamma < \arccos\frac{b}{a}\right)$, than the function $\chi_{\max}(\varphi)$ in the domain $\left(\arctan\frac{a-b\cos\gamma}{b\sin\gamma}, \frac{\pi}{2}\right)$ increasing,

$$\chi_{\max}\left(\arctan\frac{a-b\cos\gamma}{b\sin\gamma} - \gamma\right) = c, \qquad \chi_{\max}\left(\frac{\pi}{2}\right) = a.$$

If $\arctan\frac{a-b\cos\gamma}{b\sin\gamma} \le \gamma$ $\left(\text{i.e. } \cos\gamma \le \frac{b}{a} \text{ or } \gamma \ge \arccos\frac{b}{a}\right)$, than the function $\chi_{\max}(\varphi)$ in the domain $\left(\arctan\frac{a-b\cos\gamma}{b\sin\gamma}, \gamma\right)$ decreasing, while in the domain $\left(\gamma, \frac{\pi}{2}\right)$ increasing,

$$\varphi_{\min} = \gamma, \quad \chi_{\max}(\varphi_{\min}) = a\sin\gamma, \quad \chi_{\max}\left(\arctan\frac{a-b\cos\gamma}{b\sin\gamma} - \gamma\right) = c, \quad \text{and}$$

$$\chi_{\max}\left(\frac{\pi}{2}\right) = a.$$

Let $\gamma < \arccos\frac{b}{a}$. It is not difficult to verify that $c < b$, if $\cos\gamma > \frac{a}{2b}$, or $\gamma < \arccos\frac{a}{2b}$. From the other hand, $\arccos\frac{a}{2b} < \arccos\frac{b}{a}$, if $a > b\sqrt{2}$.

So, if $a > b\sqrt{2}$ and $\gamma \le \arccos\frac{a}{2b}$, than

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, b\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b\sin\gamma, c) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [c, b) \\ \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b, a] \\ \emptyset, & \text{if} \quad y > a \quad \text{or} \quad y < 0 \end{cases} \tag{4.2}$$

If $a > b\sqrt{2}$ and $\arccos\frac{a}{2b} < \gamma \le \arccos\frac{b}{a}$, we have

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, b\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b\sin\gamma, b) \\ \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b, c) \\ \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [c, a] \\ \emptyset, & \text{if} \quad y > a \quad \text{or} \quad y < 0 \end{cases} \tag{4.3}$$

When $a \le b\sqrt{2}$, for every $\gamma < \arccos\frac{b}{a}$, $c < b$, and $\Phi_{a,b}(y)$ is defined by (4.2).

Now let $\arccos\frac{b}{a} \le \gamma < \frac{\pi}{2}$. It follows from the Sine theorem for the triangles, that $c \ge a\sin\gamma$. Moreover, if $a > b\sqrt{2}$, $\arcsin\frac{b}{a} < \arccos\frac{b}{a}$, so for every $\gamma \ge \arccos\frac{b}{a}$, $a\sin\gamma \ge b$.

Therefore, in the case $a > b\sqrt{2}$ and $\arccos \frac{b}{a} \le \gamma < \arccos \frac{b}{2a}$ we have

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, b\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b\sin\gamma, b) \\ \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b, a\sin\gamma) \\ \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup & \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [a\sin\gamma, c) \\ \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [c, a) \\ \emptyset, & \text{if} \quad y > a \quad \text{or} \quad y < 0 \end{cases} \tag{4.4}$$

When $a > b\sqrt{2}$ and $\arccos \frac{b}{2a} \le \gamma < \frac{\pi}{2}$ we have

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, b\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b\sin\gamma, b) \\ \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b, a\sin\gamma) \\ \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup & \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [a\sin\gamma, a) \\ \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right), & \text{if} \quad y \in [a, c) \\ \emptyset, & \text{if} \quad y > c \quad \text{or} \quad y < 0 \end{cases} \tag{4.5}$$

Let $a \le b\sqrt{2}$ and $\arccos \frac{b}{a} \le \gamma < \frac{\pi}{2}$. If in addition $\gamma \le \arccos \frac{a}{2b}$, than we have $c < b$. So, in this case we have

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, b\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b\sin\gamma, a\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup & \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [a\sin\gamma, c) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup & \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [c, b) \\ \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b, a) \\ \emptyset, & \text{if} \quad y > a \quad \text{or} \quad y < 0 \end{cases} \tag{4.6}$$

If $a \le b\sqrt{2}$ and $\arccos\frac{a}{2b} \le \gamma \le \arcsin\frac{b}{a}$, than $a\sin\gamma \le b$ and $c \ge b,$ so we have two cases: $c \le a$ and $c > b$. By that mean we split the condition $a \le b\sqrt{2}$ into two conditions: $a \le \frac{b\sqrt{5}}{2}$ and $\frac{b\sqrt{5}}{2} \le a \le b\sqrt{2}$. Now let $a \le \frac{b\sqrt{5}}{2}$, $\arccos\frac{a}{2b} \le \gamma < \arccos\frac{b}{2a}$. In this case we have $b\sin\gamma \le a\sin\gamma \le b \le c \le a$, so

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, b\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b\sin\gamma, a\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [a\sin\gamma, b) \\ \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [b, c) \\ \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [c, a) \\ \emptyset, & \text{if} \quad y > a \quad \text{or} \quad y < 0 \end{cases} \tag{4.7}$$

When $a \le \frac{b\sqrt{5}}{2}$ and $\arccos\frac{b}{2a} \le \gamma < \arcsin\frac{b}{a}$, we have $b\sin\gamma \le a\sin\gamma \le b \le a \le c$, so

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, b\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b\sin\gamma, a\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{b\sin\gamma}{y}\right) \cup \\ \cup \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [a\sin\gamma, b) \\ \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [b, a) \\ \left(\arccos\frac{b\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right), & \text{if} \quad y \in [a, c) \\ \emptyset, & \text{if} \quad y > c \quad \text{or} \quad y < 0 \end{cases} \tag{4.8}$$

In case of $a \le \frac{b\sqrt{5}}{2}$ and $\arcsin\frac{b}{a} < \gamma < \frac{\pi}{2}$, $\Phi_{a,b}(y)$ is defined by (4.5). Let $\frac{b\sqrt{5}}{2} \le a \le b\sqrt{2}$. It is not difficult to verify that in this case when $\arccos\frac{a}{2b} \le \gamma < \arcsin\frac{b}{a}$, $\Phi_{a,b}(y)$ is defined by the formula (4.7), and when $\arcsin\frac{b}{a} \le \gamma < \arccos\frac{b}{2a}$, $\Phi_{a,b}(y)$ is defined by (4.4), while in the case of $\arccos\frac{b}{2a} \le \gamma < \frac{\pi}{2}$ it's defined by (4.5).

Consider $\gamma \ge \frac{\pi}{2}$ $(c > b,\ c > a)$. In this case the function $\frac{b\sin\gamma}{\cos\varphi}$ in the domain $\left(-\frac{\pi}{2}, \arctan\frac{a-b\cos\gamma}{b\sin\gamma}\right)$ increasing, while function $\frac{a\sin\gamma}{\cos(\varphi-\gamma)}$ in the domain

$\left(\arctan \frac{a-b\cos\gamma}{b\sin\gamma}, \frac{\pi}{2}\right)$ decreasing, so

$$\Phi_{a,b}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0,b) \\ \left(\arccos \frac{b\sin\gamma}{y}, \frac{\pi}{2}\right), & \text{if} \quad y \in [b,a) \\ \left(\arccos \frac{b\sin\gamma}{y}, \gamma - \arccos \frac{a\sin\gamma}{y}\right), & \text{if} \quad y \in [a,c) \\ \emptyset, & \text{if} \quad y > c \quad \text{or} \quad y < 0 \end{cases} \tag{4.9}$$

Finally, we obtain the following table

Cases	$\Phi_{a,b}(y)$
$a > b\sqrt{2}$ and $\gamma \leq \arccos \frac{a}{2b}$, $\quad a \leq b\sqrt{2}$ and $\gamma < \arccos \frac{b}{a}$	(4.2)
$a > b\sqrt{2}$ and $\arccos \frac{a}{2b} < \gamma \leq \arccos \frac{b}{a}$	(4.3)
$a > b\sqrt{2}$ and $\arccos \frac{b}{a} \leq \gamma < \arccos \frac{b}{2a}$, $\frac{b\sqrt{5}}{2} \leq a \leq b\sqrt{2}$ and $\arcsin \frac{b}{a} \leq \gamma < \arccos \frac{b}{2a}$	(4.4)
$a > \frac{b\sqrt{5}}{2}$ and $\arccos \frac{b}{2a} \leq \gamma < \frac{\pi}{2}$, $a \leq \frac{b\sqrt{5}}{2}$ and $\arcsin \frac{b}{a} < \gamma < \frac{\pi}{2}$	(4.5)
$a \leq b\sqrt{2}$ and $\gamma \leq \arccos \frac{a}{2b}$	(4.6)
$a \leq \frac{b\sqrt{5}}{2}$ and $\arccos \frac{a}{2b} \leq \gamma < \arccos \frac{b}{2a}$, $\frac{b\sqrt{5}}{2} \leq a \leq b\sqrt{2}$ and $\arccos \frac{a}{2b} \leq \gamma < \arcsin \frac{b}{a}$	(4.4)
$a \leq \frac{b\sqrt{5}}{2}$ and $\arccos \frac{b}{2a} \leq \gamma < \arcsin \frac{b}{a}$	(4.8)
$\frac{\pi}{2} \leq \gamma < \pi$	(4.9)

Table 1

When segments a and b have no common endpoint, we put them so, that the intersection point of lines containing a and b coincide with the origin, and segment a lie on X-axis. Denote by a' and b' the segments containing a and b, where one of the endpoints of a' and b' coincide with the origin, and another endpoint coincides with the endpoint of a and b correspondingly. Denote $a'' = a'\ a, b'' = b'\ b$. It is not difficult to verify that

$$I_{[a]\bigcap[b]}(g) = I_{[a']\bigcap[b']}(g) - I_{[a']\bigcap[b'']}(g) - I_{[a'']\bigcap[b]}(g) + I_{[a'']\bigcap[b'']}(g)\,,$$

where I is the characteristic function. Here segments a' and b', a' and b'', a'' and b', a'' and b'' have common endpoint, so we can use the table above.

In fact we obtain that for every convex polygon $\mathbb{D}$ we can find the domains $\Phi_{ij}(y)$ and calculate the integrals in (3.4).

5. CHORD LENGTH DISTRIBUTION FUNCTION FOR RHOMBUS

Consider the special case of (3.4) when $\mathbb{D}$ is a rhombus with side a and angle γ $\left(\gamma \le \frac{\pi}{2}\right)$:

$$
\begin{gathered}
F(y) = 1 - \frac{y}{2a\sin\gamma}\left[\int_{\Phi_{12}(y)} \sin\varphi\,\sin(\gamma-\varphi)\,d\varphi + \int_{\Phi_{14}(y)} \sin\varphi\,\sin(\pi-\gamma-\varphi)\,d\varphi\right] \\
+\frac{\sqrt{y^2-a^2\sin^2\gamma}}{2ay}\left[I(a\sin\gamma \le y \le 2a\sin\tfrac{\gamma}{2})\left(a(1-\cos\gamma)-\sqrt{y^2-a^2\sin^2\gamma}\right)\right. \\
+I(a\sin\gamma \le y \le a)\left(a(1-\cos\gamma)+\sqrt{y^2-a^2\sin^2\gamma}\right) \\
\left.+I(a \le y \le 2a\cos\frac{\gamma}{2})\left(a(1+\cos\gamma)-\sqrt{y^2-a^2\sin^2\gamma}\right)\right] - \frac{(a-y)^+}{a}. \qquad (5.1)
\end{gathered}
$$

From Table 1 for $\Phi_{12}(y)$ we have 2 cases: $\gamma \le \frac{\pi}{3}$, and $\frac{\pi}{3} < \gamma \le \frac{\pi}{2}$, which are defined by (4.6), (4.8) correspondingly. Let $\gamma \le \frac{\pi}{3}$. We have

$$
\Phi_{12}(y) = \begin{cases}
\left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, a\sin\gamma) \\
\left(-\frac{\pi}{2}+\gamma, -\arccos\frac{a\sin\gamma}{y}\right) \cup & \\
\cup \left(\arccos\frac{a\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup & \\
\cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [a\sin\gamma, 2a\sin\frac{\gamma}{2}) \\
\left(-\frac{\pi}{2}+\gamma, -\arccos\frac{a\sin\gamma}{y}\right) \cup & \\
\cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [2a\sin\frac{\gamma}{2}, a) \\
\emptyset, & \text{if} \quad y > a \quad \text{or} \quad y < 0
\end{cases}
$$

$\Phi_{14}(y)$ is defined by (4.9):

$$
\Phi_{14}(y) = \begin{cases}
\left(\frac{\pi}{2}-\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, a) \\
\left(\arccos\frac{a\sin\gamma}{y}, \pi-\gamma-\arccos\frac{a\sin\gamma}{y}\right), & \text{if} \quad y \in [a, 2a\cos\frac{\gamma}{2}) \\
\emptyset, & \text{if} \quad y > 2a\cos\frac{\gamma}{2} \quad \text{or} \quad y < 0
\end{cases}
$$

Calculating integrals in (5.1) we obtain

$$
F(y) = \begin{cases}
0, & \text{if} \quad y < 0 \\
\frac{y}{2a}\left[1+\left(\frac{\pi}{2}-\gamma\right)\cot\gamma\right], & \text{if} \quad y \in [0, a\sin\gamma) \\
\frac{y}{2a}\left[1-\left(\frac{\pi}{2}+\gamma-2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right]+ & \\
+\frac{\sqrt{y^2-a^2\sin^2\gamma}}{y}, & \text{if} \quad y \in [a\sin\gamma, 2a\sin\frac{\gamma}{2}) \\
\frac{y}{4a}\left[3+\left(2\arcsin\frac{a\sin\gamma}{y}-3\gamma\right)\cot\gamma\right]+ & \\
+\frac{\sqrt{y^2-a^2\sin^2\gamma}}{2y}, & \text{if} \quad y \in [2a\sin\frac{\gamma}{2}, a) \\
1+\frac{y}{4a}\left[-1+\left(\gamma-2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right]+ & \\
+\frac{\sqrt{y^2-a^2\sin^2\gamma}}{2y} & \text{if} \quad y \in [a, 2a\cos\frac{\gamma}{2}) \\
1, & \text{if} \quad y \ge 2a\cos\frac{\gamma}{2}
\end{cases} \qquad (5.2)
$$

It is not difficult to verify that function defined by (5.2) is a continuous function. Now let $\frac{\pi}{3} < \gamma \le \frac{\pi}{2}$. From (4.8) we have

$$\Phi_{12}(y) = \begin{cases} \left(-\frac{\pi}{2}+\gamma, \frac{\pi}{2}\right), & \text{if} \quad y \in [0, a\sin\gamma) \\ \left(-\frac{\pi}{2}+\gamma, -\arccos\frac{a\sin\gamma}{y}\right) \cup \\ \cup \left(\arccos\frac{a\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) \cup \\ \cup \left(\gamma + \arccos\frac{a\sin\gamma}{y}, \frac{\pi}{2}\right) & \text{if} \quad y \in [a\sin\gamma, a) \\ \left(\arccos\frac{a\sin\gamma}{y}, \gamma - \arccos\frac{a\sin\gamma}{y}\right) & \text{if} \quad y \in [a, 2a\sin\frac{\gamma}{2}) \\ \emptyset, & \text{if} \quad y > a \quad \text{or} \quad y < 0 \end{cases}$$

As in previous case $\Phi_{14}(y)$ is defined by (4.9). Finally we have

$$F(y) = \begin{cases} 0, & \text{if} \quad y < 0 \\ \frac{y}{2a}\left[1 + \left(\frac{\pi}{2}-\gamma\right)\cot\gamma\right], & \text{if} \quad y \in [0, a\sin\gamma) \\ \frac{y}{2a}\left[1 - \left(\frac{\pi}{2}+\gamma - 2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right] + \\ +\frac{\sqrt{y^2-a^2\sin^2\gamma}}{y}, & \text{if} \quad y \in [a\sin\gamma, a) \\ 1 - \frac{y}{2a}\left[1 + \left(\frac{\pi}{2}-\gamma\right)\cot\gamma\right] \\ +\frac{\sqrt{y^2-a^2\sin^2\gamma}}{y}, & \text{if} \quad y \in [a, 2a\sin\frac{\gamma}{2}) \\ 1 + \frac{y}{4a}\left[-1 + \left(\gamma - 2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right] + \\ +\frac{\sqrt{y^2-a^2\sin^2\gamma}}{2y} & \text{if} \quad y \in [2a\sin\frac{\gamma}{2}, 2a\cos\frac{\gamma}{2}) \\ 1, & \text{if} \quad y \ge 2a\cos\frac{\gamma}{2} \end{cases} \tag{5.3}$$

In (5.3) function $F(y)$ is continuous.

Thus, in terms of elementary functions we find the elementary expression of chord length distribution for rhomb. It is defined by formulae (5.2) and (5.3). If we denote by $f(y) = F'(y)$ the chord length density function, then for $\gamma \le \frac{\pi}{3}$ and for $\frac{\pi}{3} < \gamma \le \frac{\pi}{2}$ we have correspondingly

$$f(y) = \begin{cases} 0, & \text{if} \quad y < 0 \quad \text{or} \quad y \ge 2a\cos\frac{\gamma}{2} \\ \frac{1}{2a}\left[1 + \left(\frac{\pi}{2}-\gamma\right)\cot\gamma\right], & \text{if} \quad y \in [0, a\sin\gamma) \\ \frac{1}{2a}\left[1 - \left(\frac{\pi}{2}+\gamma - 2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right] + \\ +\frac{a^2\sin^2\gamma + y^2\cos\gamma}{y^2\sqrt{y^2-a^2\sin^2\gamma}}, & \text{if} \quad y \in [a\sin\gamma, 2a\sin\frac{\gamma}{2}) \\ \frac{1}{4a}\left[3 + \left(2\arcsin\frac{a\sin\gamma}{y} - 3\gamma\right)\cot\gamma\right] + \\ +\frac{a^2\sin^2\gamma - y^2\cos\gamma}{2y^2\sqrt{y^2-a^2\sin^2\gamma}}, & \text{if} \quad y \in [2a\sin\frac{\gamma}{2}, a) \\ \frac{1}{4a}\left[-1 + \left(\gamma - 2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right] + \\ +\frac{a^2\sin^2\gamma + y^2\cos\gamma}{2y^2\sqrt{y^2-a^2\sin^2\gamma}}, & \text{if} \quad y \in [a, 2a\cos\frac{\gamma}{2}) \end{cases} \tag{5.4}$$

$$f(y) = \begin{cases} 0, & \text{if} \quad y < 0 \text{ or } y \geq 2a\cos\frac{\gamma}{2} \\ \frac{1}{2a}\left[1 + \left(\frac{\pi}{2} - \gamma\right)\cot\gamma\right], & \text{if} \quad y \in [0, a\sin\gamma) \\ \frac{1}{2a}\left[1 - \left(\frac{\pi}{2} + \gamma - 2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right] + & \\ +\frac{a^2\sin^2\gamma + y^2\cos\gamma}{y^2\sqrt{y^2 - a^2\sin^2\gamma}}, & \text{if} \quad y \in [a\sin\gamma, a) \\ -\frac{1}{2a}\left[1 + \left(\frac{\pi}{2} - \gamma\right)\cot\gamma\right] + & \\ +\frac{a\sin^2\gamma}{y^2\sqrt{y^2 - a^2\sin^2\gamma}}, & \text{if} \quad y \in [a, 2a\sin\frac{\gamma}{2}) \\ \frac{1}{4a}\left[-1 + \left(\gamma - 2\arcsin\frac{a\sin\gamma}{y}\right)\cot\gamma\right] + & \\ +\frac{a^2\sin^2\gamma + y^2\cos\gamma}{2y^2\sqrt{y^2 - a^2\sin^2\gamma}}, & \text{if} \quad y \in [2a\sin\frac{\gamma}{2}, 2a\cos\frac{\gamma}{2}), \end{cases} \tag{5.5}$$

The graphs of chord length distribution and density function for rhombus with side $a = 1$ and angles $\gamma = \frac{\pi}{6}$ and $\gamma = \frac{5\pi}{12}$ see in Figures 1 and 2.

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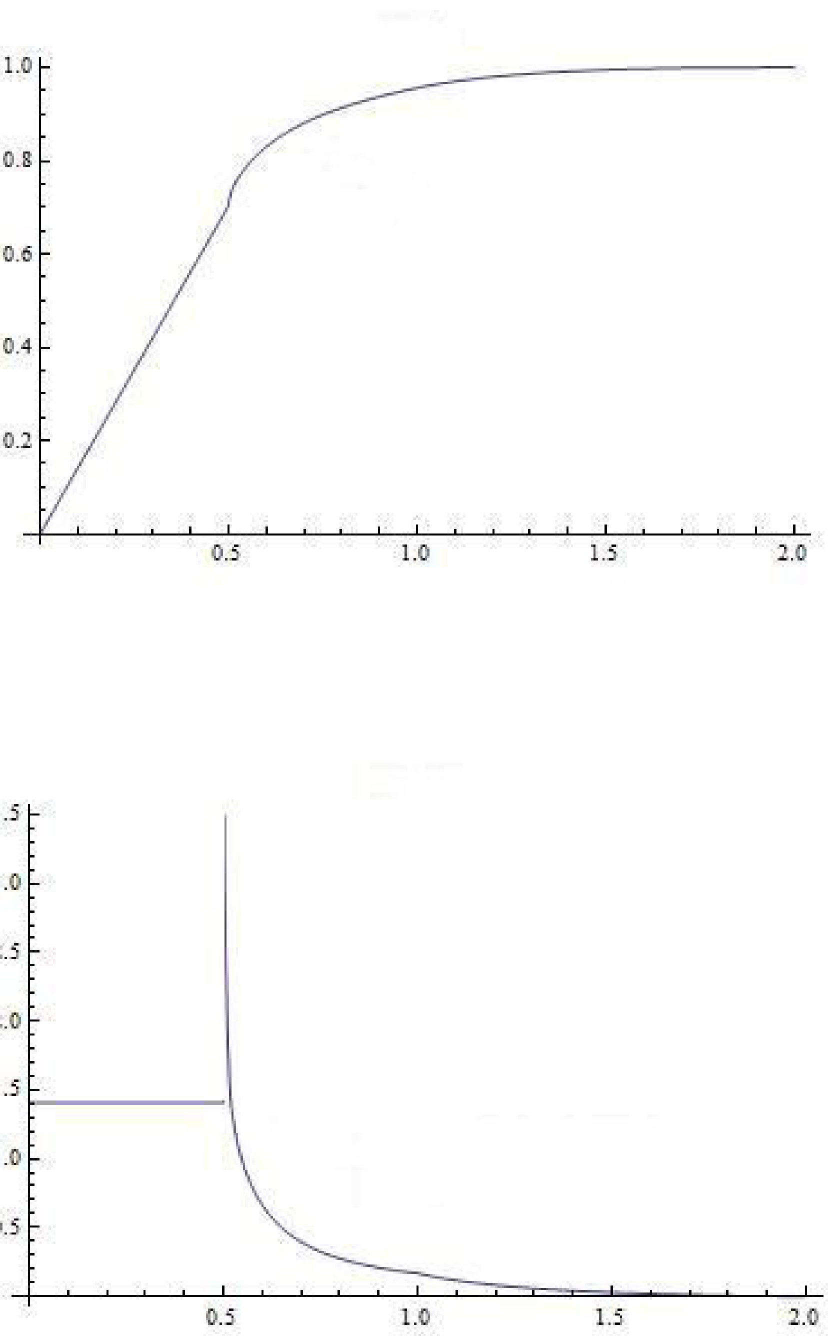

Fig. 1. Chord length distribution and density functions for rhombus with $a = 1$ and $\gamma = \frac{\pi}{6}$.

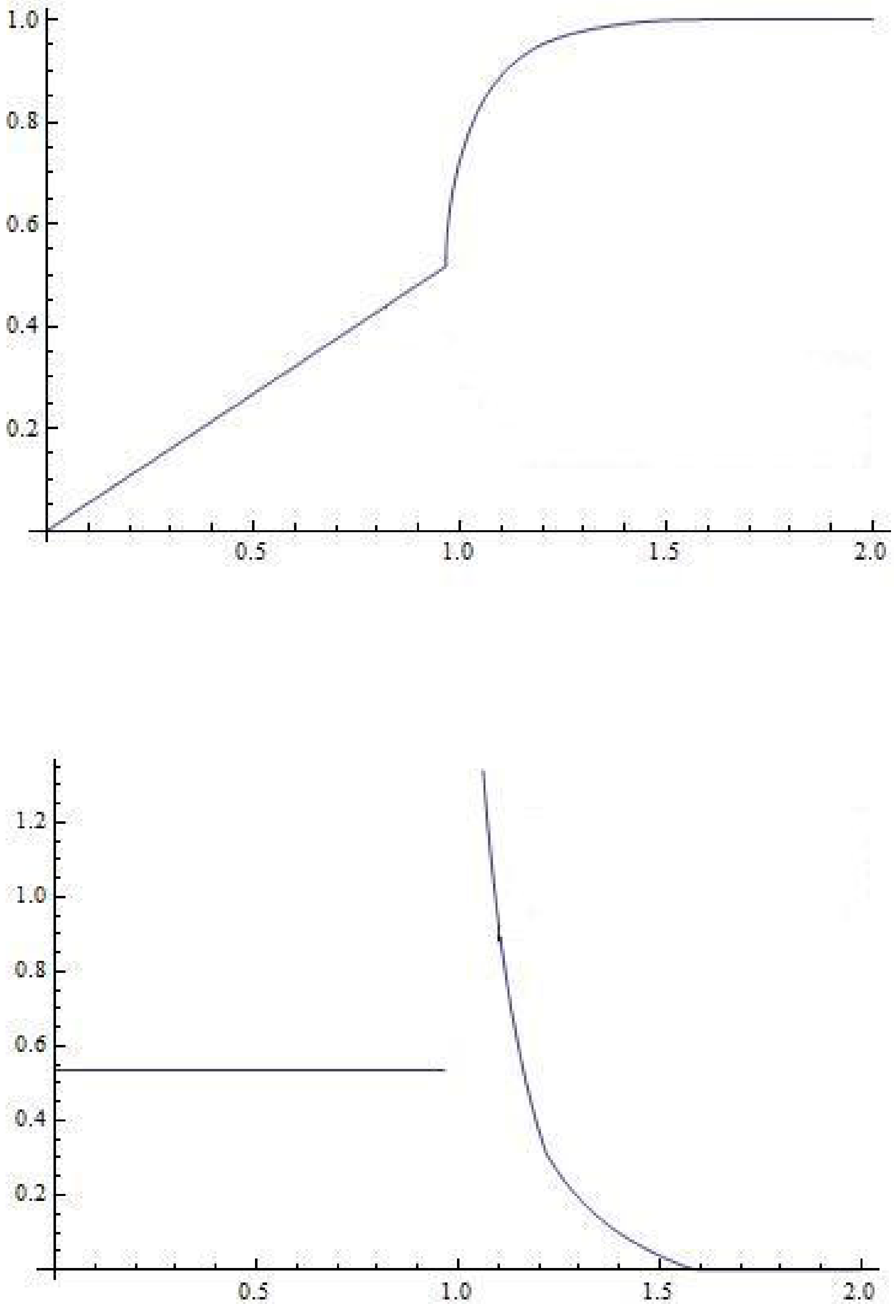

Fig. 2. Chord length distribution and density functions for rhombus with $a = 1$ and $\gamma = \frac{5\pi}{12}$.