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THE CENTROID OF SOME GENERALIZED PEDAL CONFIGURATIONS

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The main goal of this paper is to give possible generalizations, analogues of the following property:

If M_1, M_2 and M_3 are the orthogonal projections of a point M to the sides A_1A_2, A_2A_3 and A_3A_1 of an equilateral triangle $A_1A_2A_3$, then the centroid of the triangle $M_1M_2M_3$ is the midpoint of the segment OM, where O is the center of the triangle $A_1A_2A_3$.

In the first step we extend this property to regular n-gons, regular tetrahedrons and regular n simplices. In the second part we give a general affine version for triangles and simplices.

It is also our objective to analyze the possibility of using such properties in teaching problem solving strategies for students and mathematics teachers. Theorem 2, 2.3, 2.5, 2.10 and Conjecture 1 and 2 were discovered/rediscovered during a training course for mathematics teachers.

Keywords: pedal polygons; pedal simplices; centroids.

1. Introduction

In this note we analyze some possible generalizations of the following property:

Theorem 1.1. *If we denote by M_1, M_2 and M_3 the orthogonal projections of a point M to the sides A_1A_2, A_2A_3 and A_3A_1 of an equilateral triangle $A_1A_2A_3$, then the centroid G of the triangle $M_1M_2M_3$ is the midpoint of the line segment OM, where O is the center of triangle $A_1A_2A_3$.*

This problem was studied on a teacher training course at the Babeş-Bolyai University from the perspective of the inquiry based learning. The starting point was not only the problem, but also a solution of this problem using complex numbers. The aim of the instructors was to support the participants (secondary school mathematics teachers) in experimenting, developing and proving generalizations of this property. This approach was used in order to prove that inquiry based learning

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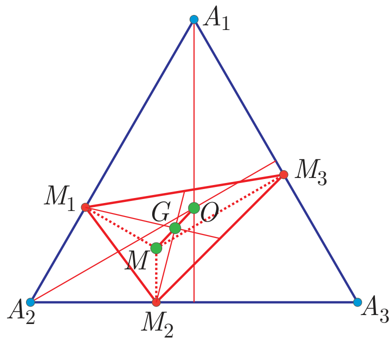

Fig. 1. The centroid of a podar triangle relative to an equilateral triangle

can also be used in problem solving activities and in the framework of the existing curricula. The main tools for experimenting were dynamic geometry softwares (Geogebra, Geonext and Cabri) and the teachers were working in groups. In the first round each group had to present some ideas about the possible generalizations, in the second round each group had to formulate conjectures and had to experiment his own conjectures and in the final round they had to prove the conjectures which seemed to be valid.

2. Conjectures and proofs

As a first step we can replace the equilateral triangle with something more general: an arbitrary triangle, a regular polygon, a regular tetrahedron or a regular simplex. Or we can use some more general projections in constructing the pedal triangle. In order to formulate some more general properties we recall a proof for theorem 1.1.

Proof. We assume M is in the plane of the triangle $A_1A_2A_3$, and the circumscribed circle is of radius 1. Let's consider O as the origin of the coordinate system and OA_1 as the OX axis. The complex numbers corresponding to the vertices A_1, A_2 and A_3 are

$$a_1 = \varepsilon, \quad a_2 = \varepsilon^2 \text{ and } a_3 = \varepsilon^3,$$

where $\varepsilon^3 = 1$ and $\varepsilon \neq 1$. Due to our assumptions, m_1 is easy to find, because it's real part is $-\frac{1}{2}$ and it's imaginary part is the same as m's. So

$$m_1 = -\frac{1}{2} + \frac{m - \overline{m}}{2}.$$

To calculate m_2 we'll use a counterclockwise rotation of angle $\alpha = \frac{4\pi}{3}$. This rotation transforms the point M_2 to the orthogonal projection of the point $Q(\varepsilon^2 \cdot m)$ to the side A_1A_2. Hence we have

$$m_2 \cdot \varepsilon^2 = -\frac{1}{2} + \frac{\varepsilon^2 \cdot m - \left(\overline{\varepsilon^2 \cdot m}\right)}{2},$$

and so

$$m_2 = -\frac{1}{2} \cdot \varepsilon + \frac{m - \varepsilon^2 \cdot \overline{m}}{2}.$$

By the same argument we deduce

$$m_3 = -\frac{1}{2} \cdot \varepsilon^2 + \frac{m - \varepsilon \cdot \overline{m}}{2}.$$

From these relations we get

$$\frac{m_1 + m_2 + m_3}{3} = \frac{m}{2},$$

which expresses the desired property. □

The above presented proof suggests that something similar holds also for a regular n-gon. A little experience with a dynamic geometry software helps us to formulate the following property:

Theorem 2.1. *If we denote by M_i ($1 \leq i \leq n$) the orthogonal projections of a point M to the sides A_iA_{i+1} ($1 \leq i \leq n$ and $A_{n+1} = A_1$) of a regular n-gon $A_1A_2A_3 \ldots A_n$ and by O the center of the polygon, then the centroid of the n-gon $M_1M_1M_2 \ldots M_n$ is the midpoint of the line segment OM.*

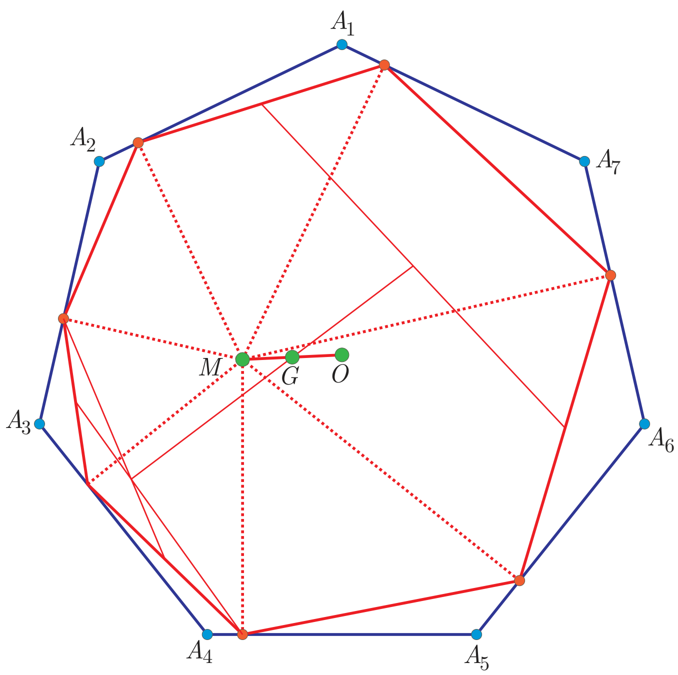

Fig. 2. The centroid of a podar polygon relative to a regular n-gon

Proof. First we consider the case when n is odd. The vertices of the regular n-gon are represented by

$$a_j = \varepsilon^j = \cos\frac{2j\pi}{n} + \sin\frac{2j\pi}{n}, \quad 1 \le j \le n.$$

If $n = 2k+1$, we have $A_k A_{k+1} \parallel OY$ and hence

$$m_k = \cos\frac{2k\pi}{2k+1} + \frac{m - \overline{m}}{2}.$$

Using rotations again we can deduce

$$m_j \cdot \varepsilon^{k-j} = \cos\frac{2k\pi}{2k+1} + \frac{m \cdot \varepsilon^{k-j} - \overline{(m \cdot \varepsilon^{k-j})}}{2}$$

for $1 \le j \le 2k+1$. But $\overline{\varepsilon} = \varepsilon^{2k} = \varepsilon^{-1}$ and hence we have the following relations:

$$m_j = \varepsilon^{k+j+1} \cos\frac{2k\pi}{2k+1} + \frac{m - \varepsilon^{2j+1} \cdot \overline{m}}{2} \quad \text{for} \quad 1 \le j \le 2k+1.$$

These relations and the $\sum\limits_{v=0}^{n-1} \varepsilon^v = 0$ equality imply

$$\frac{1}{n} \cdot \sum_{j=1}^{n} m_j = \frac{m}{2}.$$

If n is even ($n = 2k$) we take

$$a_j = z_0 \varepsilon^j = z_0 \left(\cos\frac{2j\pi}{n} + i \cdot \sin\frac{2j\pi}{n} \right), \quad 1 \le j \le n,$$

where

$$z_0 = \cos\frac{\pi}{2k} + i \cdot \sin\frac{\pi}{2k}$$

and we obtain

$$m_{k-1} = \cos\frac{(2k-1)\pi}{2k} + \frac{m - \overline{m}}{2},$$

$$m_j \cdot \varepsilon^{k-j-1} = \cos\frac{(2k-1)\pi}{2k} + \frac{m \cdot \varepsilon^{k-j-1} - \overline{(m \cdot \varepsilon^{k-j-1})}}{2}$$

for $1 \le j \le 2k$. These relations imply $\frac{1}{n} \cdot \sum\limits_{j=1}^{n} m_j = \frac{m}{2}$ and hence the proof is completed. □

Remark 2.2. *If $n = 2k$, the points corresponding to the complex numbers $\frac{m_j + m_{j+k}}{2}$ are on the axis of symmetry parallel to $A_j A_{j+1}$. So the relation*

$$\frac{1}{n} \cdot \sum_{j=1}^{n} m_j = \frac{1}{k} \cdot \sum_{j=1}^{k} \frac{m_j + m_{j+k}}{2} = \frac{m}{2}$$

means that the centroid of the k-gon determined by the projections of M to the these axis of symmetry is the midpoint of OM. In this case the property can be viewed as theorem 2.1 with a degenerated regular k-gon (which degenerates into the center of symmetry).

The previous remark suggests that a similar property holds if the points $M_1, M_2, \ldots, M_n$ are the projections of M to the axis of symmetry. To explore this possibility and to formulate a proper conjecture we can construct figures using a geometry software. Finally we obtain the following property:

Theorem 2.3. *The centroid of the n-gon determined by the orthogonal projections of a point M to the axis of symmetry of a regular n-gon is the midpoint of OM (O is the center of symmetry).*

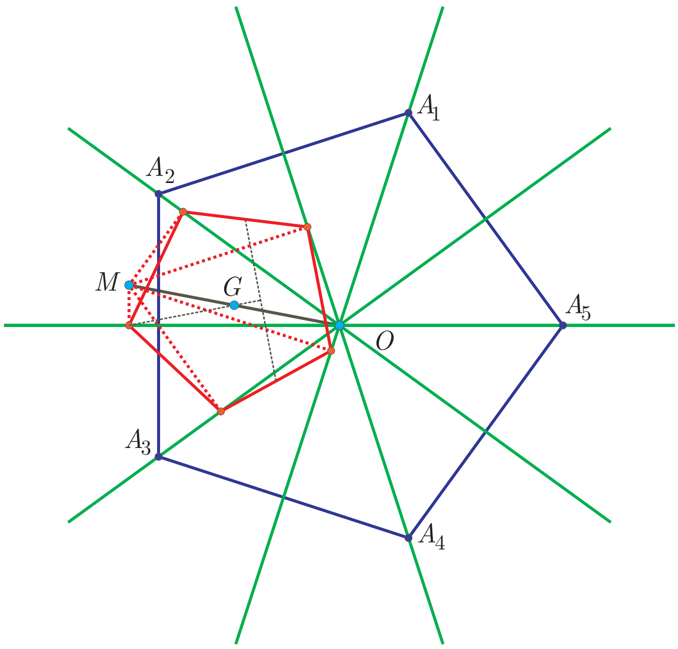

Fig. 3. The centroid of a podar polygon relative to the symmetry axes of a regular n-gon

Proof. The method and the calculations are almost the same. We can assume that one of the symmetry axis is the OY axes, so the real part of the projections affix is 0 while the imaginary part is the same as the imaginary part of the projected point's affix. This implies that if we repeat all the calculations instead of $\cos\frac{(2k-1)\pi}{2k}$ or $\cos\frac{2k\pi}{2k+1}$ we have 0. □

Remark 2.4. *All groups discovered/rediscovered independently theorem 2.1 while theorem 2.3 was observed only by one group when they analyzed their proof.*

In order to generalize the previous results to higher dimensions as a first step we analyze the case of a regular tetrahedron.

Theorem 2.5. *Let $A_1A_2A_3A_4$ be a regular tetrahedron with center O, M_i $(1 \leq i \leq 4)$the orthogonal projections of a point M to it's faces and Q_i $(1 \leq i \leq 6)$ the projections to it's sides. The following two statements are true:*

a) The centroid G_2 of the tetrahedron $M_1M_2M_3M_4$ is on OM and satisfies $\frac{OG_2}{OM} = \frac{2}{3}$.
b) The centroid G_1 of the system $Q_1Q_2Q_3Q_4Q_5Q_6$ is on OM and satisfies $\frac{OG_1}{OM} = \frac{1}{3}$.

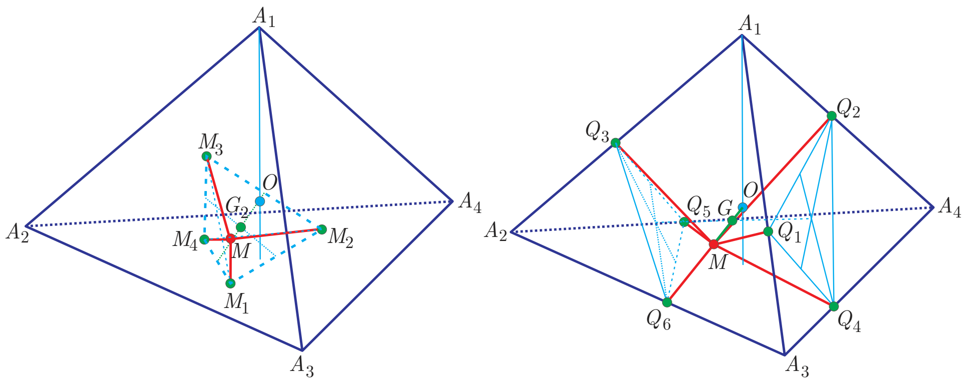

Fig. 4. The centroid of a podar tetrahedron relative to the faces and the sides of a regular tetrahedron

Proof. First we prove that the two statements are equivalent. Due to theorem 1.1. we have (see figure 4):

$$\frac{\overrightarrow{OO_1} + \overrightarrow{OM_1}}{2} = \frac{\overrightarrow{OQ_4} + \overrightarrow{OQ_5} + \overrightarrow{OQ_6}}{3},$$

$$\frac{\overrightarrow{OO_2} + \overrightarrow{OM_2}}{2} = \frac{\overrightarrow{OQ_4} + \overrightarrow{OQ_1} + \overrightarrow{OQ_2}}{3},$$

$$\frac{\overrightarrow{OO_3} + \overrightarrow{OM_3}}{2} = \frac{\overrightarrow{OQ_3} + \overrightarrow{OQ_5} + \overrightarrow{OQ_2}}{3}$$

and

$$\frac{\overrightarrow{OO_4} + \overrightarrow{OM_4}}{2} = \frac{\overrightarrow{OQ_1} + \overrightarrow{OQ_3} + \overrightarrow{OQ_6}}{3},$$

(where $Q_1 \in A_1A_3, Q_2 \in A_1A_4, Q_3 \in A_1A_2, Q_4 \in A_4A_3, Q_5 \in A_2A_4, Q_6 \in A_2A_3$, $\overrightarrow{AB}$ means the vector from A to B, and O_i are the centroids of the faces). From these equalities we deduce

$$\frac{1}{2} \cdot \sum_{i=1}^{4} \overrightarrow{OM_i} = \frac{2}{3} \cdot \sum_{i=1}^{6} \overrightarrow{OQ_i}$$

(because $\frac{1}{2} \cdot \sum\limits_{i=1}^{4} \overrightarrow{OO_i} = 0$) and so

$$\frac{1}{2} \cdot \overrightarrow{OG_2} = \overrightarrow{OG_1}.$$

This relation implies the equivalence of the two statements.

In order to prove the first statement we consider a regular tetrahedron with vertices $A_1\left(0, 0, \frac{\sqrt{6}}{3}\right)$, $A_2\left(-\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0\right)$, $A_3\left(\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0\right)$, $A_4\left(0, \frac{\sqrt{3}}{3}, 0\right)$ and we calculate the coordinates of the points M_i. The equations of the faces are:

$$\begin{aligned} A_1A_2A_3: &\quad -2\sqrt{3}y + \sqrt{3}z - \sqrt{2} = 0, \\ A_1A_2A_4: &\quad 3\sqrt{2}x - \sqrt{6}y - \sqrt{3}z + \sqrt{2} = 0, \\ A_1A_3A_4: &\quad 3\sqrt{2}x + \sqrt{6}y + \sqrt{3}z - \sqrt{2} = 0 \quad \text{and} \\ A_3A_2A_4: &\quad z = 0 \end{aligned}$$

Using the formula

$$x = x_0 - A \cdot \frac{A \cdot x_0 + B \cdot y_0 + C \cdot z_0 + D}{A^2 + B^2 + C^2},$$

which gives the x coordinate of the orthogonal projection of the point (x_0, y_0, z_0) to the plane with equation $A \cdot x + B \cdot y + C \cdot z + D = 0$ we obtain:

$$x_1 = x_4 = x_0,$$

$$x_2 = x_0 - 3\sqrt{2} \cdot \frac{3\sqrt{2} \cdot x_0 + \sqrt{6} \cdot y_0 + \sqrt{3} \cdot z_0 - \sqrt{2}}{27} \quad \text{and}$$

$$x_3 = x_0 - 3\sqrt{2} \cdot \frac{3\sqrt{2} \cdot x_0 - \sqrt{6} \cdot y_0 - \sqrt{3} \cdot z_0 + \sqrt{2}}{27}.$$

From these equalities we can easily deduce the relation

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{2x_0}{3}.$$

A similar calculation shows the same relation for the y and the z coordinates, hence G_2 lies on OP and satisfies the desired equality. This completes the proof. □

Based on this property we can formulate the following conjecture for the n-dimensional euclidian space:

Conjecture 1. *If we denote by G_k the centroid of the system determined by the orthogonal projections of a point M to the k-dimensional faces of a regular n-simplex $A_1A_2A_3...A_nA_{n+1}$, than $G_k \in OM$ and satisfies $\frac{OG_k}{OM} = \frac{k}{n}$, $1 \leq k \leq n-1$, where O is the center of the simplex.*

Remark 2.6. *Theorem 2.5 was discovered/rediscovered by all the groups while conjecture 1 was formulated (without proof) only by one group. After an analysis of the proof of theorem 2.5 all groups formulated the necessity of a different approach in attacking the higher dimensional problem.*

This conjecture can be proved using the same ideas as in the proof of theorem 2.5, but the calculations are more complicated. In order to simplify the proof of this conjecture first we try to give an affine version of the 2 dimensional property and then by extending this affine version to higher dimensions we find a property which is more general than conjecture 1 and admits a simpler proof. For this we need to observe that in a regular polygon (polyhedra or simplex) the segment joining the centroid O of the polygon with the midpoint O_i of a side (the centroid of a face) is perpendicular to this side (face). Hence the construction of the orthogonal projection of M to a side d can be viewed as the construction of a projection in the direction OO_i. This can be done for an arbitrary triangle or even for an arbitrary polygon, so we can formulate the following conjectures:

Conjecture 2. *In the triangle $A_1A_2A_3$ $O_1 \in A_2A_3$, $O_2 \in A_3A_1$ and $O_3 \in A_1A_2$ are the midpoints of the sides and $A_1O_1 \cap A_2O_2 \cap A_3O_3 = \{O\}$. If for an arbitrary point M we consider the points $M_1 \in A_2A_3$, $M_2 \in A_3A_1$ and $M_3 \in A_1A_2$ such that $MM_1 || OO_1$, $MM_2 || OO_2$ and $MM_3 || OO_3$, then the centroid of the triangle $M_1M_2M_3$ is the midpoint of the segment OM.*

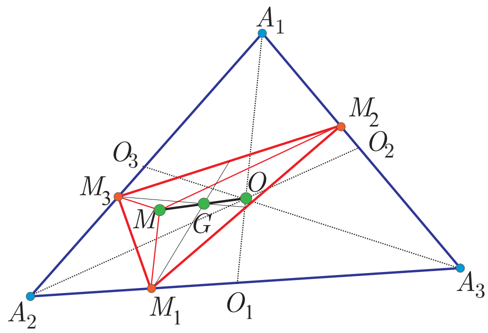

Fig. 5. The centroid of a generalized podar triangle relative to an arbitrary triangle

Conjecture 3. *Denote by $O_1, O_2, \ldots, O_n$ the midpoints of the sides A_1A_2, $A_2A_3, \ldots,$ A_nA_1 in the polygon $A_1A_2 \ldots A_n$ and by O the centroid of the poly-*

gon. If for an arbitrary point M we consider the points $M_i \in A_iA_{i+1}$, $1 \le i \le n$ such that $MM_i || OO_i$, $1 \le i \le n$, then the centroid of the polygon $M_1M_2 \dots M_n$ is the midpoint of the segment OM.

Using a dynamic geometry software we can explore the validity of these conjectures. Conjecture 2 seems to be true but unfortunately Conjecture 3 is not true for all polygons.

Remark 2.7. *It is an open question to assure conditions for which conjecture 3 becomes a theorem. One of the groups observed that conjecture 3 becomes a theorem for $n = 4$ if $A_1A_2A_3A_4$ is a trapezoid.*

Proof of conjecture 2. If we denote by the corresponding small letter the position vector of each point, we have the following relations: $o = \frac{1}{3}(a_1 + a_2 + a_3)$, $o_1 = \frac{1}{2}(a_2 + a_3)$, $o_2 = \frac{1}{2}(a_3 + a_1)$ and $o_3 = \frac{1}{2}(a_1 + a_2)$. M is an arbitrary point in the plane so there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$m = \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3$$

and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Since $M_1 \in A_2A_3$ there exists $\lambda_1 \in \mathbb{R}$ with the property $m_1 = \lambda_1 a_2 + (1 - \lambda_1)a_3$, hence the condition $MM_1 || OO_1$ can be expressed as

$$m - m_1 = c(o - o_1)$$

with $c \in \mathbb{R}$. This implies

$$\alpha_1 a_1 + (\alpha_2 - \lambda_1)\, a_2 + (\alpha_3 - 1 + \lambda_1)\, a_3 = c\left(\frac{1}{3}a_1 - \frac{1}{6}a_2 - \frac{1}{6}a_3\right).$$

If the origin of the position vectors is not in the same plane as the vertices of the initial triangle, than a_1, a_2 and a_3 are linearly independent vectors, so we obtain the following system

$$\begin{cases} \alpha_1 &= \frac{1}{3}c \\ \alpha_2 - \lambda_1 &= -\frac{c}{6} \\ \alpha_3 - 1 + \lambda_1 &= -\frac{c}{6} \end{cases}$$

and so

$$m_1 = m - 3\alpha_1(o - o_1) = \left(\alpha_2 + \frac{1}{2}\alpha_1\right) a_2 + \left(\alpha_3 + \frac{1}{2}\alpha_1\right) a_3.$$

Using a similar argument we obtain

$$m_2 = \left(\alpha_3 + \frac{1}{2}\alpha_2\right) a_3 + \left(\alpha_1 + \frac{1}{2}\alpha_2\right) a_1 \text{ and } m_3 = \left(\alpha_1 + \frac{1}{2}\alpha_3\right) a_1 + \left(\alpha_2 + \frac{1}{2}\alpha_3\right) a_2$$

so

$$g = \frac{1}{3}\left(m_1 + m_2 + m_3\right) = \frac{1}{2}(m + o).$$ □

Analyzing theorem 2.5 and conjecture 2 we can formulate the following property:

Theorem 2.8. *(Zsolt Szilágyi, Szilárd András) Consider the n-simplex $A_0 \dots A_n$, an arbitrary point M and O the centroid of the simplex. Denote by $O_{i_0 \dots i_k}$ the centroid of the face $A_{i_0} \dots A_{i_k}$ and by $M_{i_0 \dots i_k}$ the intersection of the face $A_{i_0} \dots A_{i_k}$ with the line drawn through M and parallel to $O_{i_0 \dots i_k} A_{i_{k+1} \dots A_{i_n}}$. If $G_k^{'}$ denotes the centroid of the system $M_{i_0 \dots i_k}$, where k is fixed and $i_0 \dots i_k$ takes all possible values, then for each $k \in \{1, 2, 3 \dots, n-1\}$ the points $M, O,\ G_k^{'}$ are on the same line and*

$$\frac{OG_k^{'}}{OM} = \frac{k}{n}.$$

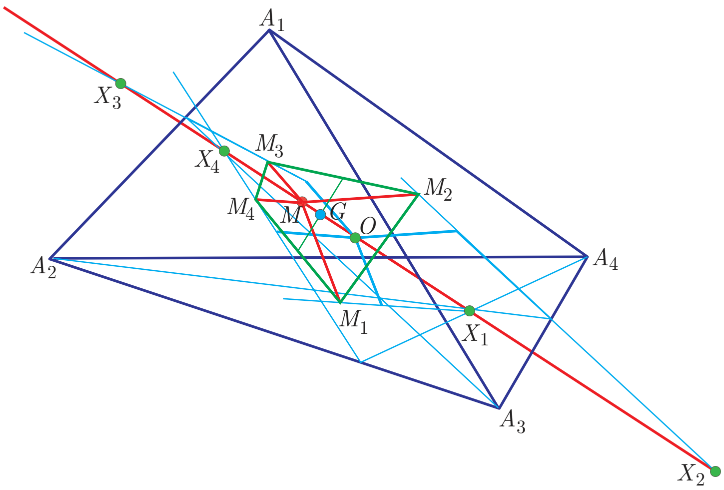

Fig. 6. The centroid of a podar tetrahedron relative to an arbitrary tetrahedron

Proof. Denote by the corresponding small letters the position vectors (in $\mathbb{R}^n$) of the points. M is an arbitrary point in $\mathbb{R}^n$, so there exist $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ such that $\sum\limits_{i=0}^{n} \alpha_i = 1$ and

$$m = \sum_{i=0}^{n} \alpha_i a_i.$$

If $\overline{m} := m_{i_0 \dots i_k} = \sum\limits_{j=0}^{k} c_j a_{i_j}$ (with $\sum\limits_{j=0}^{k} c_j = 1$) is the intersection of the face $A_{i_0} \dots A_{i_k}$ with the parallel line to $O_{i_0 \dots i_k} A_{i_{k+1}} \dots A_{i_n}$, and $o_{i_0 \dots i_k} = \frac{1}{k+1} \sum\limits_{j=0}^{k} a_{i_j}$ is the centroid of the face $A_{i_0} \dots A_{i_k}$, then the condition of parallelism is:

$$m - \overline{m} = c \left(\sum_{j=k+1}^{n} \lambda_j (a_{i_j} - o_{i_0 \dots i_k}) \right),$$

where $\lambda_j \in \mathbb{R}$, for all $j \in \{k+1, \ldots, n\}$ and $\sum\limits_{j=k+1}^{n} \lambda_j = 1$. This can be written as

$$\sum_{j=0}^{k}(\alpha_{i_j} - c_j)a_{i_j} + \sum_{j=k+1}^{n} \alpha_{i_j} a_{i_j} = c\left(\sum_{j=k+1}^{n} \lambda_j a_{i_j} - \frac{1}{k+1}\sum_{j=0}^{k} a_{i_j}\right),$$

so we deduce $\lambda_j = \frac{\alpha_{i_j}}{c}$, for $k+1 \le j \le n$, and $c_j = \alpha_{i_j} + \frac{c}{k+1}$, for $0 \le j \le k$. From these relations we obtain $c = \sum\limits_{j=k+1}^{n} \alpha_{i_j}$, and so

$$\overline{m} = m_{i_0 \ldots i_k} = \sum_{j=0}^{k}\left[\alpha_{i_j} + \frac{1}{k+1}(\alpha_{i_{k+1}} + \ldots + \alpha_{i_n})\right] a_{i_j}.$$

The number of the k dimensional faces in an n-simplex is C_{n+1}^{k+1}, hence

$$\begin{aligned} g_k^{'} &= \frac{1}{C_{n+1}^{k+1}} \sum_{i \in C_{n+1,k+1}} m_{i_0 \ldots i_k} \\ &= \frac{1}{C_{n+1}^{k+1}} \sum_{i \in C_{n+1,k+1}} \left(\sum_{j=0}^{k}\left[\alpha_{i_j} + \frac{1}{k+1}(\alpha_{i_{k+1}} + \ldots + \alpha_{i_n})\right] a_{i_j}\right), \end{aligned}$$

where $C_{n+1,k+1}$ denotes the set of all combinations of order $(k+1)$ formed from the set $\{0, 1, \ldots, n+1\}$. The number of combinations i for which $l \in \{i_0, \ldots, i_k\}$, where l is a fixed index, is C_n^k while the number of combinations i for which $l \in \{i_0, \ldots, i_k\}$ and $j \in \{i_{k+1}, \ldots, i_n\}$ is C_{n-1}^k. This implies

$$\begin{aligned} g_k^{'} &= \frac{1}{C_{n+1}^{k+1}} \sum_{l=0}^{n}\left[C_n^k \alpha_l + \sum_{j \neq l} \frac{C_{n-1}^k}{k+1}\alpha_j\right] a_l \\ &= \frac{1}{C_{n+1}^{k+1}} \sum_{l=0}^{n}\left[C_n^k \alpha_l + \frac{C_{n-1}^k}{k+1}(1-\alpha_l)\right] a_l \\ &= \sum_{l=0}^{n}\left(\frac{k+1}{n+1}\alpha_l + \frac{n-k}{n(n+1)}(1-\alpha_l)\right) a_l \\ &= \sum_{l=0}^{n}\left(\frac{k}{n}\alpha_l + \frac{n-k}{n(n+1)}\right) a_l. \end{aligned}$$

The centroid is $o = \frac{1}{n+1}\sum\limits_{i=0}^{n} a_i$, so

$$\begin{aligned} \overrightarrow{OG_k^{'}} &= \sum_{l=0}^{n}\left(-\frac{1}{n+1} + \frac{k}{n}\alpha_l + \frac{n-k}{n(n+1)}\right) a_l \\ &= \sum_{l=0}^{n}\left(-\frac{k}{n(n+1)} + \frac{k}{n}\alpha_l\right) a_l. \end{aligned} \tag{1}$$

But $\overrightarrow{OM} = \sum_{l=0}^{n} \left(\alpha_l - \frac{1}{n+1}\right) a_l$, hence $\overrightarrow{OG_k} = \frac{k}{n}\overrightarrow{OM}$, which completes the proof. $\square$

Remark 2.9. *Theorem 2.8 proves Conjecture 1, because in a regular simplex $O_{i_0 \dots i_k} A_{i_{k+1}} \dots A_{i_n}$ is orthogonal to the face $A_{i_0} \dots A_{i_k}$. At the training course this theorem and it's proof was presented by the instructors.*

Examining theorem 2 the following natural question arise: can we replace the midpoints of the faces by other points? The following theorem answers this question.

Theorem 2.10. *Consider a triangle $A_1A_2A_3$ and the real numbers w_1, w_2, w_3 with $w_1 + w_2 + w_3 = 1$. Let O be the point with barycentric coordinates (w_1, w_2, w_3) and M an arbitrary point in the plane of the triangle. If we construct the points $M_1 \in A_2A_3$, $M_2 \in A_3A_1$ and $M_3 \in A_1A_2$ such that $MM_1 || OA_1$, $MM_2 || OA_3$ and $MM_3 || OA_3$, then the centroid of the triangle $M_1M_2M_3$ coincides with the centroid of the triangle MOP, where the barycentric coordinates of P relative to $M_1M_2M_3$ are (w_1, w_2, w_3).*

Remark 2.11. *If $w_1 = w_2 = w_3$, then P is the centroid of $M_1M_2M_3$, so the above theorem reduces to conjecture 2.*

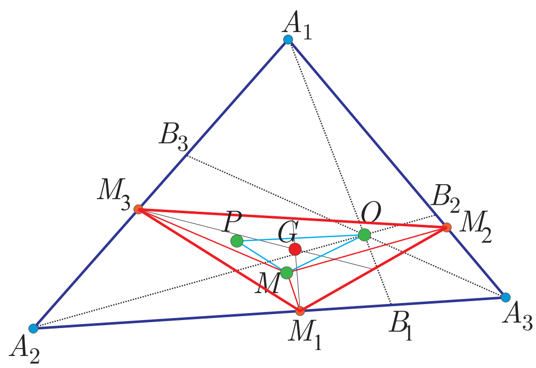

Fig. 7. The centroid of a generalized podar triangle relative to an arbitrary triangle

Proof. If $\gamma_1, \gamma_2, \gamma_3$ denote the barycentric coordinates of M, then

$$m = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$$

and

$$o = w_1 a_1 + w_2 a_2 + w_3 a_3,$$

where a_1, a_2, a_3 are the position vectors of the vertices and m, o the corresponding position vectors of the points M and O. $M_1 \in A_2A_3$, so there exists $\lambda_1 \in \mathbb{R}$

such that $m_1 = \lambda_1 a_2 + (1 - \lambda_1)a_3$. The condition $MM_1 || OA_1$ can be expressed as $m_1 - m = c(o - a_1)$, with some $c \in \mathbb{R}$. This implies

$$\lambda_1 a_2 + (1 - \lambda_1)a_3 - \gamma_1 a_1 - \gamma_2 a_2 - \gamma_3 a_3 = c(w_1 a_1 + w_2 a_2 + w_3 a_3 - a_1) \quad (2)$$

If we suppose that the starting point of the position vectors is outside the plane $A_1A_2A_3$, then a_1, a_2, a_3 are linearly independent, hence (2) implies $-\gamma_1 = c(w_1 - 1)$, $\lambda_1 - \gamma_2 = cw_2$ and $1 - \lambda_1 - \gamma_3 = cw_3$. From these relations we obtain

$$m_1 = \left(\gamma_2 + \gamma_1 \frac{w_2}{w_2 + w_3}\right) a_2 + \left(\gamma_3 + \gamma_1 \frac{w_3}{w_2 + w_3}\right) a_3. \quad (3)$$

By a similar reasoning we deduce

$$m_2 = \left(\gamma_3 + \gamma_2 \frac{w_3}{w_1 + w_3}\right) a_3 + \left(\gamma_1 + \gamma_2 \frac{w_1}{w_1 + w_3}\right) a_1, \quad (4)$$

$$m_3 = \left(\gamma_1 + \gamma_3 \frac{w_1}{w_2 + w_1}\right) a_1 + \left(\gamma_2 + \gamma_3 \frac{w_2}{w_2 + w_1}\right) a_2. \quad (5)$$

From (3), (4) and (5) we deduce

$$m_1(1 - w_1) + m_2(1 - w_2) + m_3(1 - w_3) = m + o,$$

thus

$$\frac{1}{3}(m_1 + m_2 + m_3) = \frac{1}{3}(m + o + p),$$

which is the desired property. □

Remark 2.12. *Theorem 2.10 can be extended also to higher dimensional simpleces.*

Remark 2.13. *It would be interesting to find an affine version which generalizes all the previous properties (including theorem 2.1 too).*

3. Concluding remarks

- From scientific point of view the training course was fruitful because the participants discovered new properties of the studied configurations and they developed new generalizations, which clarify some aspects of the problem.
- Remark 2.7 and remark 2.13 shows that interesting questions arose, they can constitute a starting point for further study.
- From a teaching point of view the main aim of the training course was achieved, the participants got familiar with the inquiry based methods, they realized that a deeper inquiry of the problems can constitute a solid motivational base for further individual study. They also perceived that in many cases the understanding of the background implicitly requires a deeper inquiry and this can be done even in the traditional framework and context.

- Some participants pointed out that the major inference of this training course was that they realized the importance of simultaneous alternatives in solving a problem and the importance of selecting the tools they use in designing the proofs of well known properties.
- The participants emphasized that a major task in preparing and designing an inquiry based lesson is the selection of a sufficiently rich problem.

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