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SUBCLASSES OF P-VALENT STARLIKE FUNCTIONS DEFINED BY USING CERTAIN FRACTIONAL DERIVATIVE OPERATOR

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In this paper we introduce two subclasses of p-valent starlike functions in the open unit disk by using fractional derivative operator. We obtain coefficient inequalities and distortion theorems for functions belonging to these subclasses. Further results include distortion theorems (involving the generalized fractional derivative operator). The radii of convexity for functions belonging to these subclasses are also studied.

Keywords: p-valent functions; starlike functions; convex functions; fractional derivative operators.

1. Introduction

Fractional calculus operators have recently found interesting applications in the theory of analytic functions. The classical definition of fractional calculus and its other generalizations have fruitfully been applied in obtaining, the characterization properties, coefficient estimates and distortion inequalities for various subclasses of analytic functions. For numerous references on the subject, one may refer to the works by Srivastava and Owa (1989) and Srivastava and Owa (1992). See also Owa (1978); Srivastava and Owa (1987); Raina and Srivastava (1996) and Raina and Nahar (2002).

Let $A(p)$ denote the class of functions defined by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} \qquad (p \in \mathbb{N}) \tag{1.1}$$

which are analytic and p-valent in the unit disk $\mathcal{U} = \{z: |z| < 1\}$. Also denote $T(p)$, the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n}\, z^{p+n} \qquad \left(a_{p+n} \geq 0, \quad p \in \mathbb{N}\right) \tag{1.2}$$

Two interesting subclasses $T^*(\alpha,\beta,\gamma)$ and $C(\alpha,\beta,\gamma)$ of univalent starlike functions with negative coefficients in the open unit disk $\mathcal{U}$ were introduced by Srivastava and Owa (1991a). In fact, these classes become the subclasses of the class $K(\alpha,\beta)$ which was introduced by Gupta (1984) when the function $f(z)$ is univalent with negative coefficients. Using the results of Srivastava and Owa (1991a), Srivastava and Owa (1991b) have obtained several distortion theorems involving fractional derivatives and fractional integrals of functions belonging to the classes $T^*(\alpha,\beta,\gamma)$ and $C(\alpha,\beta,\gamma)$.

Recently, Aouf and Hossen (2006) have generalized the results of Srivastava and Owa (1991a) to the case of certain p-valent functions with negative coefficients.

With these points in view, by making use of a certain fractional derivative operator, we introduce new subclasses $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ (defined below) of p-valent starlike functions with negative coefficients.

This paper is organized as follows: Section 2 gives preliminary details and definitions of p-valent starlike functions, p-valent convex functions and fractional derivative operators. In Section 3 we describe coefficient inequalities for the functions belonging to the subclasses $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Section 4 considers the distortion properties. Its further distortion properties (involving the generalized fractional derivative operator) are discussed in section 5. Finally, in section 6 we determine the radii of convexity for functions belonging to these subclasses of p-valent starlike functions.

2. Preliminaries And Definitions

A function $f(z) \in A(p)$ is called p-valent starlike of order α if $f(z)$ satisfies the conditions

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \tag{2.1}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2p\pi \tag{2.2}$$

for $0 \le \alpha < p,\ p \in \mathbb{N}$, and $z \in \mathcal{U}$. We denote by $S^*(p,\alpha)$ the class of all p-valent starlike functions of order α. Also a function $f(z) \in A(p)$ is called p-valent convex of order α if $f(z)$ satisfies the following conditions:

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \tag{2.3}$$

and

$$\int_0^{2\pi} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta = 2p\pi \tag{2.4}$$

for $0 \leq \alpha < p$, $p \in \mathbb{N}$, and $z \in \mathcal{U}$. We denote by $K(p,\alpha)$ the class of all p-valent convex functions of order α. We note that

$$f(z) \in K(p,\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(p,\alpha) \tag{2.5}$$

for $0 \leq \alpha < p$.

The class $S^*(p,\alpha)$ was introduced by Patil and Thakare (1983), and the class $K(p,\alpha)$ was introduced by Owa (1985). Also, we denote by $T^*(p,\alpha)$ and $C(p,\alpha)$ the classes obtained by taking intersections, respectively, of the classes $S^*(p,\alpha)$ and $K(p,\alpha)$ with $T(p)$; that is

$$T^*(p,\alpha) = S^*(p,\alpha) \cap T(p)$$

and

$$C(p,\alpha) = K(p,\alpha) \cap T(p)$$

The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ were introduced by Owa (1985). In particular, the classes $T^*(1,\alpha) = T^*(\alpha)$ and $C(1,\alpha) = C(\alpha)$ when $p = 1$, were studied by Silverman (1975).

Let ${}_2F_1(a,b;c;z)$ be the Gauss hypergeometric function defined, for $z \in \mathcal{U}$ by; see Srivastava and Karlsson (1985).

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n\, z^n}{(c)_n\, n!} \tag{2.6}$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & , \quad n = 0 \\ \lambda(\lambda+1)(\lambda+2)\ldots.(\lambda+n-1) & , \quad n \in \mathbb{N} \end{cases} \tag{2.7}$$
$$(\lambda \neq 0,-1,-2,\ldots)$$

Now we recall the following definitions of fractional derivative operators, adopted for working in classes of analytic functions in complex plane which were used by Owa (1978); Srivastava and Owa (1987); Srivastava and Owa (1991b); Raina and Nahar (2002); Raina and Srivastava (1996); see also, Srivastava and Owa (1989) and Srivastava and Owa (1992) as follows.

Definition 2.1. *The fractional derivative of order λ is defined, for a function $f(z)$, by*

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{\lambda}} d\xi \tag{2.8}$$

where $0 \leq \lambda < 1$ *is analytic function in a simply- connected region of the z-plane containing the origin, and the multiplicity of* $(z-\xi)^{-\lambda}$ *is removed by requiring log* $(z-\xi)$ *to be real when* $z-\xi > 0$.

Definition 2.2. *Let* $0 \leq \lambda < 1$, *and* $\mu, \eta \in \mathbb{R}$. *Then, in terms of the familiar Gauss's hypergeometric function* ${}_2F_1$ *, the generalized fractional derivative operator* $J_{0,Z}^{\lambda,\mu,\eta}$ *is*

$$J_{0,Z}^{\lambda,\mu,\eta} f(z) = \frac{d}{dz}\left(\frac{z^{\lambda-\mu}}{\Gamma(1-\lambda)}\int_0^z (z-\xi)^{-\lambda} f(\xi)\, {}_2F_1\left(\mu-\lambda, 1-\eta; 1-\lambda; 1-\frac{\xi}{z}\right) d\xi\right) \quad (2.9)$$

where f(z) is analytic function in a simply- connected region of the z-plane containing the origin with the order $f(z) = O(|z|^{\varepsilon})$, $z \to 0$, *where* $\varepsilon > max\{0, \mu-\eta\} - 1$, *and the multiplicity of* $(z-\xi)^{-\lambda}$ *is removed by requiring log* $(z-\xi)$ *to be real when* $z-\xi > 0$.

Notice that

$$J_{0,Z}^{\lambda,\lambda,\eta} f(z) = D_z^{\lambda} f(z), \qquad 0 \leq \lambda < 1 \qquad (2.10)$$

Srivastava and Owa (1987) have established various distortion theorems for the fractional calculus of functions $f(z)$ belonging to the classes $T^*(p,\alpha)$ and $C(p,\alpha)$.

We now define the following classes of p-valent starlike functions based on fractional derivative operator.

Definition 2.3. *The function* $f(z) \in T(p)$ *is said to be in the class* $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ *if*

$$\left| \frac{\dfrac{z\left(\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} - p}{\dfrac{z\left(\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)} + p - 2\beta} \right| < \gamma \qquad (z \in \mathcal{U}) \qquad (2.11)$$

$(\lambda \geq 0, \mu < p+1\,;\ \eta > \max(\lambda,\mu) - p - 1\,;\ 0 \leq \alpha < p;\ 0 \leq \beta < p;\ 0 < \gamma \leq 1;\ p \in \mathbb{N})$

For the function

$$g(z) = z^p - \sum_{n=1}^{\infty} b_{p+n}\, z^{p+n} \qquad \left(b_{p+n} \geq 0\,, p \in \mathbb{N}\right) \qquad (2.12)$$

belonging to $T^*(p,\alpha)$. *Dented by* $M_{0,z}^{\lambda,\mu,\eta} f(z)$ *the modification of the fractional derivative operator which is defined in terms of* $J_{0,Z}^{\lambda,\mu,\eta}$ *as follows:*

$$\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z) = \phi_p(\lambda,\mu,\eta)\, z^{\mu}\, J_{0,Z}^{\lambda,\mu,\eta} f(z) \qquad (2.13)$$

with

$$\phi_p(\lambda,\mu,\eta) = \frac{\Gamma(1-\mu+p)\Gamma(1+\eta-\lambda+p)}{\Gamma(1+p)\Gamma(1+\eta-\mu+p)} \qquad (2.14)$$

Further, if $f(z) \in T(p)$ *satisfies the condition (2.11) for* $g(z) \in C(p,\alpha)$, *we say that* $f(z) \in C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$.

The above-defined classes $T^{*}_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ are of special interest and they contain many well-known classes of analytic functions. In particular, in view of (2.13), we find that

$$\mathrm{M}^{0,0,\eta}_{0,z} f(z) = f(z) \tag{2.15}$$

Thus, for $\lambda = \mu = 0$, we have

$$T^{*}_{0,0,\eta}(p,\alpha,\beta,\gamma) = T^{*}(p,\alpha,\beta,\gamma)$$

and

$$C_{0,0,\eta}(p,\alpha,\beta,\gamma) = C(p,\alpha,\beta,\gamma)$$

where $T^{*}(p,\alpha,\beta,\gamma)$ and $C(p,\alpha,\beta,\gamma)$ are precisely the subclasses of p-valent starlike functions which were studied by Aouf and Hossen (2006).

Furthermore, for $\lambda = \mu = 0$ and $p = 1$, we obtain

$$T^{*}_{0,0,\eta}(1,\alpha,\beta,\gamma) = T^{*}(\alpha,\beta,\gamma)$$

and

$$C_{0,0,\eta}(1,\alpha,\beta,\gamma) = C(\alpha,\beta,\gamma)$$

where $T^{*}(\alpha,\beta,\gamma)$ and $C(\alpha,\beta,\gamma)$ are the subclasses of starlike functions which were studied by Srivastava and Owa (1991a) as well as by Srivastava and Owa (1991b).

In order to prove our results we shall need the following lemmas for the classes $T^{*}(p,\alpha)$ and $C(p,\alpha)$ due to Owa (1985):

Lemma 2.4. *Let the function* $g(z)$ *defined by (2.12). Then* $g(z)$ *is in the class* $T^{*}(p,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}(p+n-\alpha)b_{p+n} \leq (p-\alpha) \tag{2.16}$$

Lemma 2.5. *Let the function* $g(z)$ *defined by (2.12). Then* $g(z)$ *is in the class* $C(p,\alpha)$ *if and only if*

$$\sum_{n=1}^{\infty}(p+n)(p+n-\alpha)b_{p+n} \leq p(p-\alpha) \tag{2.17}$$

We mention to the following known result which shall be used in the sequel (Raina and Nahar (2002); see also Raina and Srivastava (1996)).

Lemma 2.6. *Let* $\lambda,\mu,\eta \in \mathbb{R}$, *such that* $\lambda \geq 0$ *and* $k > max\{0,\mu-\eta\}-1$ *then*

$$J_{0,z}^{\lambda,\mu,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\mu+\eta+1)}{\Gamma(k-\mu+1)\Gamma(k-\lambda+\eta+1)} z^{k-\mu} \tag{2.18}$$

3. Coefficient Inequalities

Theorem 3.1. *Let the function $f(z)$ be defined by (1.2). If $f(z)$ belongs to the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, then*

$$\sum_{n=1}^{\infty} \delta_n(\lambda,\mu,\eta,p)(p+n)\,(1+\gamma)a_{p+n} - \frac{(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{p+n-\alpha} \leq 2\gamma(p-\beta) \tag{3.1}$$

where

$$\delta_n(\lambda,\mu,\eta,p) = \frac{\phi_p(\lambda,\mu,\eta)}{\phi_{p+n}(\lambda,\mu,\eta)} = \frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n} \tag{3.2}$$

and $\phi_p(\lambda,\mu,\eta)$ is given by (2.14).

Proof. Applying Lemma 2.6, we have from (1.2) and (2.13) that

$$\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z) = z^p - \sum_{n=1}^{\infty} \frac{(1+p)_n(1+\eta-\mu+p)_n}{(1-\mu+p)_n(1+\eta-\lambda+p)_n}\, a_{p+n}\, z^{p+n}$$

Since $f(z) \in T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, there exist a function $g(z)$ belonging to the class $T^*(p,\alpha)$ such that

$$\left| \frac{z\left(\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)\right)' - pg(z)}{z\left(\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)\right)' + (p-2\beta)g(z)} \right| < \gamma, \quad z \in \mathcal{U} \tag{3.3}$$

It follows from (3.3) that

$$\mathrm{Re}\left\{ \frac{\sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)\,a_{p+n} - p\,b_{p+n}]\; z^n}{2(p-\beta) - \sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} + (p-2\beta)b_{p+n}]z^n} \right\} < \gamma \tag{3.4}$$

Choosing values of z on the real axis so that $\frac{z\left(\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)\right)'}{g(z)}$ is real, and letting $z \to 1^-$ through real axis, we have

$$\sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)\,a_{p+n} - p\,b_{p+n}] \leq$$

$$\gamma\left\{2(p-\beta) - \sum_{n=1}^{\infty}[\delta_n(\lambda,\mu,\eta,p)(p+n)a_{p+n} + (p-2\beta)b_{p+n}]\right\}$$

or, equivalently,

$$\sum_{n=1}^{\infty}\{\delta_n(\lambda,\mu,\eta,p)(p+n)\,(1+\gamma)a_{p+n} - [p(1-\gamma)+2\gamma\beta]\, b_{p+n}\} \leq 2\gamma(p-\beta) \quad (3.5)$$

Note that, by using Lemma 2.4, $g(z) \in T^*(p,\alpha)$ implies

$$b_{p+n} \leq \frac{p-\alpha}{p+n-\alpha} \quad (3.6)$$

Making use of (3.6) in (3.5), we complete the proof of Theorem 3.1. □

Corollary 3.2. *Let the function* $f(z)$ *be defined by (1.2) be in the class* $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. *Then*

$$a_{p+n} \leq \frac{2\gamma(p-\beta)(p+n-\alpha)+(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)(1+\gamma)(p+n-\alpha)} \quad (3.7)$$

where $\delta_n(\lambda,\mu,\eta,p)$ *is given by (3.2). The result (3.7) is sharp for a function of the form:*

$$f(z) = z^p - \frac{2\gamma(p-\beta)(p+n-\alpha)+(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)(1+\gamma)(p+n-\alpha)} z^{p+n} \quad (3.8)$$

with respect to

$$g(z) = z^p - \frac{p-\alpha}{p+n-\alpha} z^{p+n}, \qquad n \geq 1 \quad (3.9)$$

Remark 1. Letting $p = 1$, $\lambda = \mu = 0$, and $\alpha = 0$ in Corollary 3.2, we obtain a result was proved by [Gupta (1984), Theorem 3].

In a similar manner, Lemma 2.5 can be used to prove the following theorem:

Theorem 3.3. *Let the function* $f(z)$ *be defined by (1.2) be in the class* $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. *Then*

$$\sum_{n=1}^{\infty}\delta_n(\lambda,\mu,\eta,p)(p+n)\,(1+\gamma)a_{p+n} - \frac{p(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{(p+n)(p+n-\alpha)} \leq 2\gamma(p-\beta) \quad (3.10)$$

where $\delta_n(\lambda,\mu,\eta,p)$ *is given by (3.2).*

Corollary 3.4. *Let the function* $f(z)$ *be defined by (1.2) be in the class* $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. *Then*

$$a_{p+n} \leq \frac{2\gamma(p-\beta)(p+n)(p+n-\alpha)+p(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)^2(1+\gamma)(p+n-\alpha)} \quad (3.11)$$

where $\delta_n(\lambda,\mu,\eta,p)$ *is given by (3.2). The result (3.11) is sharp for a function of the form:*

$$f(z) = z^p - \frac{2\gamma(p-\beta)(p+n)(p+n-\alpha) + p(p-\alpha)[p(1-\gamma)+2\gamma\beta]}{\delta_n(\lambda,\mu,\eta,p)(p+n)^2(1+\gamma)(p+n-\alpha)} z^{p+n} \tag{3.12}$$

with respect to

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+n)(p+n-\alpha)} z^{p+n}, \qquad n \geq 1 \tag{3.13}$$

4. Distortion Properties

Next, we state and prove results concerning distortion properties of $f(z)$ belonging to the classes $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$.

Theorem 4.1. *Let* $\lambda,\mu,\eta \in \mathbb{R}$ *such that*

$$\lambda \geq 0, \mu < p+1\,, \;\; \eta \geq \lambda\left(1 - \frac{p+2}{\mu}\right) \text{ and } p \in \mathbb{N} \tag{4.1}$$

Also, let $f(z)$ *defined by (1.2) be in the class* $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. *Then*

$$|f(z)| \geq |z|^p - A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)\,|z|^{p+1}, \tag{4.2}$$

$$|f(z)| \leq |z|^p + A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^{p+1}, \tag{4.3}$$

$$|f'(z)| \geq p|z|^{p-1} - (\mathrm{p}+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p, \tag{4.4}$$

and

$$|f'(z)| \leq p|z|^{p-1} + (\mathrm{p}+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p, \tag{4.5}$$

for $z \in \mathcal{U}$, *provided that* $0 \leq \alpha < p, 0 \leq \beta < p$ *and* $0 < \gamma \leq 1$, *where*

$$A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma) = \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)^2(1+\gamma)(p+1-\alpha)} \tag{4.6}$$

The estimates for $|f(z)|$ *and* $|f'(z)|$ *are sharp.*

Proof. We observe that the function $\delta_n(\lambda,\mu,\eta,p)$ defined by (3.2) satisfy the inequality $\delta_n(\lambda,\mu,\eta,p) \leq \delta_{n+1}(\lambda,\mu,\eta,p)$, $\forall n \in \mathbb{N}$, provided that $\eta \geq \lambda\left(1 - \frac{p+2}{\mu}\right)$. Thereby, showing that $\delta_n(\lambda,\mu,\eta,p)$ is non-decreasing. Thus under conditions stated in (4.1), we have

$$0 < \frac{(1+p)(1+p+\eta-\mu)}{(1+p-\mu)(1+p+\eta-\lambda)} = \delta_1(\lambda,\mu,\eta,p) \leq \delta_n(\lambda,\mu,\eta,p), \qquad \forall n \in \mathbb{N} \tag{4.7}$$

For $f(z) \in T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, (3.5) implies

$$\delta_1(\lambda,\mu,\eta,p)(p+1)(1+\gamma)\sum_{n=1}^{\infty} a_{p+n} - [p(1-\gamma)+2\gamma\beta]\sum_{n=1}^{\infty} b_{p+n} \leq 2\gamma(p-\beta) \tag{4.8}$$

For $g(z) \in T^*(p,\alpha)$, Lemma 2.4 yields

$$\sum_{n=1}^{\infty} b_{p+n} \leq \frac{p-\alpha}{p+1-\alpha} \tag{4.9}$$

So that (4.8) reduces to

$$\sum_{n=1}^{\infty} a_{p+n} \leq \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)^2(1+\gamma)(p+1-\alpha)}$$

$$= A_{\lambda,\mu,\eta}(\,p,\alpha,\beta,\gamma) \tag{4.10}$$

Consequently,

$$|f(z)| \geq |z|^p - |z|^{p+1}\sum_{n=1}^{\infty} a_{p+n} \tag{4.11}$$

and

$$|f(z)| \leq |z|^p + |z|^{p+1}\sum_{n=1}^{\infty} a_{p+\mathrm{n}} \tag{4.12}$$

On using (4.11), (4.12) and (4.10), we easily arrive at the desired results (4.2) and (4.3).

Furthermore, we note from (3.5) that

$$\delta_1(\lambda,\mu,\eta,p)(1+\gamma)\sum_{n=1}^{\infty}(p+n)a_{p+n} - [p(1-\gamma)+2\gamma\beta]\sum_{n=1}^{\infty} b_{p+n} \leq 2\gamma(p-\beta) \tag{4.13}$$

which in view of (4.9), becomes

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \leq \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)(1+\gamma)(p+1-\alpha)}$$

$$= (p+1)A_{\lambda,\mu,\eta}(\,p,\alpha,\beta,\gamma) \tag{4.14}$$

Thus, we have

$$|f'(z)| \geq p|z|^{p-1} - |z|^p\sum_{n=1}^{\infty}(p+n)a_{p+n} \tag{4.15}$$

and

$$|f'(z)| \leq p|z|^{p-1} + |z|^p\sum_{n=1}^{\infty}(p+n)\,a_{p+n} \tag{4.16}$$

On using (4.15), (4.16) and (4.14), we arrive at the desired results (4.4) and (4.5).

Finally, we can prove that the estimates for $|f(z)|$ and $|f'(z)|$ are sharp by taking the function

$$f(z) = z^p - \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)^2(1+\gamma)(p+1-\alpha)} z^{p+1} \qquad (4.17)$$

with respect to

$$g(z) = z^p - \frac{p-\alpha}{p+1-\alpha} z^{p+1}, \qquad n \geq 1 \qquad (4.18)$$

This completes the proof of Theorem 4.1. □

Corollary 4.2. *Let the function $f(z)$ be defined by (1.2) be in the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < r_1$, where*

$$r_1 = 1 - \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)\}}{(1+p+\eta-\mu)(p+1)^2(1+\gamma)(p+1-\alpha)} \qquad (4.19)$$

The result is sharp with the extremal function defined by (4.17).

Remark 2. Letting $p = 1$, $\lambda = \mu = 0$, and $\alpha = 0$ in Theorem 4.1, we obtain a result was proved by [Gupta (1984), Theorem 4].

Theorem 4.3. *Under the conditions stated in (4.1), let the function $f(z)$ defined by (1.2) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then*

$$|f(z)| \geq |z|^p - B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^{p+1}, \qquad (4.20)$$

$$|f(z)| \leq |z|^p + B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^{p+1}, \qquad (4.21)$$

$$|f'(z)| \geq p|z|^{p-1} - (p+1)B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p, \qquad (4.22)$$

and

$$|f'(z)| \leq p|z|^{p-1} + (p+1)B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)|z|^p, \qquad (4.23)$$

for $z \in \mathcal{U}$, provided that $0 \leq \alpha < p, 0 \leq \beta < p$ and $0 < \gamma \leq 1$, where

$$B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma) = \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)(p+1)(p+1-\alpha)+p(p-\alpha)[p(1-\gamma)+2\gamma\beta]\}}{(1+p+\eta-\mu)(p+1)^3(1+\gamma)(p+1-\alpha)} \qquad (4.24)$$

The estimates for $|f(z)|$ and $|f'(z)|$ are sharp.

Proof. By using Lemma 2.5, we have

$$\sum_{n=1}^{\infty} b_{p+n} \leq \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} \qquad (4.25)$$

since $g(z) \in C(p,\alpha)$, the assertions (4.20), (4.21), (4.22) and (4.23) of Theorem 4.3 follow if we apply (4.25) to (3.5).

The estimates for $|f(z)|$ and $|f'(z)|$ are attained by the function

$$f(z) = z^p - \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)(p+1)(p+1-\alpha)+p(p-\alpha)[p(1-\gamma)+2\gamma\beta]\}}{(1+p+\eta-\mu)(p+1)^3(1+\gamma)(p+1-\alpha)} z^{p+1} \qquad (4.26)$$

with respect to

$$g(z) = z^p - \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)} z^{p+1}, \qquad n \geq 1 \tag{4.27}$$

This completes the proof of Theorem 4.3. □

Corollary 4.4. *Let the function $f(z)$ be defined by (1.2) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then the unit disk $\mathcal{U}$ is mapped onto a domain that contains the disk $|w| < r_2$, where*

$$r_2 = 1 - \frac{(1+p-\mu)(1+p+\eta-\lambda)\{2\gamma(p-\beta)(p+1)(p+1-\alpha)+p(p-\alpha)[p(1-\gamma)+2\gamma\beta]\}}{(1+p+\eta-\mu)(p+1)^3(1+\gamma)(p+1-\alpha)} \tag{4.28}$$

The result is sharp with the extremal function defined by (4.26).

5. Further Distortion Properties

We next prove two further distortion theorems involving generalized fractional derivative operator $J_{0,Z}^{\lambda,\mu,\eta}$.

Theorem 5.1. *Let $\lambda \geq 0;\ \mu < p+1; \eta > max(\lambda,\mu) - p - 1,$ and $p \in \mathbb{N}$. Also let the function $f(z)$ be defined by (1.2) be in the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$.Then*

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \geq \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)}\left\{1 - \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}|z|\right\} \tag{5.1}$$

and

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \leq \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)}\left\{1 + \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}|z|\right\} \tag{5.2}$$

for $z \in \mathcal{U}$, and $\phi_p(\lambda,\mu,\eta)$ is given by (2.14). The results (5.1) and (5.2) are sharp.

Proof. Consider the function $\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)$ defined by (2.13). With the aid of (4.7) and (4.14) we find that

$$\left|\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)\right| \geq |z|^p - \delta_1(\lambda,\mu,\eta,p)|z|^{p+1} \sum_{n=1}^{\infty} (p+n)a_{p+n}$$

$$\geq |z|^p - \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}|z|^{p+1} \tag{5.3}$$

and

$$\left|\mathrm{M}_{0,z}^{\lambda,\mu,\eta} f(z)\right| \le |z|^p + \delta_1(\lambda,\mu,\eta,p)|z|^{p+1} \sum_{n=1}^{\infty} (p+n)a_{p+n}$$
$$\le |z|^p + \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)} |z|^{p+1} \qquad (5.4)$$

which yields the inequality (5.1) and (5.2) of Theorem 5.1.

Finally, by taking the function $f(z)$ defined by

$$J_{0,Z}^{\lambda,\mu,\eta} f(z) = \frac{z^{p-\mu}}{\phi_p(\lambda,\mu,\eta)} \left\{1 - \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)} \mathrm{z}\right\} \qquad (5.5)$$

The results (5.1) and (5.2) are easily seen to be sharp. □

Corollary 5.2. *Let the function* $f(z)$ *defined by (1.2) be in the class* $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. *Then* $J_{0,Z}^{\lambda,\mu,\eta} f(z)$ *is included in a disk with its centre at the origin and radius* r_3 *given by*

$$r_3 = \frac{1}{\phi_p(\lambda,\mu,\eta)} \left\{1 + \frac{2\gamma(p-\beta)+p(p-\alpha)(1+\gamma)}{(1+\gamma)(p+1-\alpha)}\right\} \qquad (5.6)$$

Similarly we can establish the following result:

Theorem 5.3. *Let* $\lambda \ge 0$; $\mu < p+1; \eta > max(\lambda,\mu) - p - 1$, *and* $p \in \mathbb{N}$, *and let the function* $f(z)$ *be defined by (1.2) be in the class* $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. *Then*

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \ge \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)} \left\{1 - \frac{2\gamma(p-\beta)(\mathrm{p}+1)(p+1-\alpha)+p(p-\alpha)[\mathrm{p}(1-\gamma)+2\gamma\beta]}{(1+\mathrm{p})(1+\gamma)(p+1-\alpha)} |\mathrm{z}|\right\} \qquad (5.7)$$

and

$$\left|J_{0,Z}^{\lambda,\mu,\eta} f(z)\right| \le \frac{|z|^{p-\mu}}{\phi_p(\lambda,\mu,\eta)} \left\{1 + \frac{2\gamma(p-\beta)(\mathrm{p}+1)(p+1-\alpha)+p(p-\alpha)[\mathrm{p}(1-\gamma)+2\gamma\beta]}{(1+\mathrm{p})(1+\gamma)(p+1-\alpha)} |\mathrm{z}|\right\} \qquad (5.8)$$

for $z \in \mathcal{U}$, *and* $\phi_p(\lambda,\mu,\eta)$ *is given by (2.14). The results (5.7) and (5.8) are sharp.*

Corollary 5.4. *Let the function* $f(z)$ *defined by (1.2) be in the class* $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. *Then* $J_{0,Z}^{\lambda,\mu,\eta} f(z)$ *is included in a disk with its centre at the origin and radius* r_4 *given by*

$$r_4 = \frac{1}{\phi_p(\lambda,\mu,\eta)} \left\{1 + \frac{2\gamma(p-\beta)(\mathrm{p}+1)(p+1-\alpha)+p(p-\alpha)[\mathrm{p}(1-\gamma)+2\beta\gamma]}{(1+\mathrm{p})(1+\gamma)(p+1-\alpha)}\right\} \qquad (5.9)$$

Remark 3. Letting $p = 1, \mu = \lambda$ and using the relationship (2.10) in Theorem 5.1, Corollary 5.2, Theorem 5.3, and Corollary 5.4, we obtain the results which were proved by [Srivastava and Owa (1991b), Theorem 5, Corollary 3, Theorem 6, and Corollary 4, respectively].

6. Convexity Of Functions In $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ And $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$

In view of Lemma 2.4, we know that the function $f(z)$ defined by (1.2) is p-valent starlike in the unit disk $\mathcal{U}$ if and only if

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \le p \tag{6.1}$$

for $f(z) \in T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, we find from (3.5) and (4.9) that

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \le (p+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma) \le p \tag{6.2}$$

where $A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ is defined by (4.6). Furthermore, $f(z) \in C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$, we have

$$\sum_{n=1}^{\infty}(p+n)a_{p+n} \le (p+1)B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma) \le p \tag{6.3}$$

where $B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ is defined by (4.24). Thus we observe that $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ are subclasses of p-valent starlike functions. Naturally, therefore, we are interested in finding the radii of convexity for functions in $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. We first state:

Theorem 6.1. *Let the function $f(z)$ defined by (1.2) be in the class $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then $f(z)$ is p-valent convex in the disk $|z| < r_5$, where*

$$r_5 = \inf_{n\in\mathbb{N}}\left\{\frac{p^2}{(p+1)(p+n)\,A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}\right\}^{1/n} \tag{6.4}$$

and $A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ is given by (4.6). The result is sharp.

Proof. It suffices to prove

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| \le p \qquad (|z| < r_5) \tag{6.5}$$

Indeed we have

$$\left|1+\frac{zf''(z)}{f'(z)}-p\right| = \left|\frac{-\sum_{n=1}^{\infty} n(p+n)a_{p+n}z^n}{p-\sum_{n=1}^{\infty}(p+\mathrm{n})a_{p+n}z^n}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty} n(p+n)\, a_{p+n}|z|^n}{p - \sum_{n=1}^{\infty}(p+n)a_{p+n}|z|^n} \tag{6.6}$$

Hence (6.5) is true if

$$\sum_{n=1}^{\infty} n(p+n)\, a_{p+n}|z|^n \leq p^2 - \sum_{n=1}^{\infty} p(p+n)a_{p+n}|z|^n \tag{6.7}$$

that is, if

$$\sum_{n=1}^{\infty} (p+n)^2\, a_{p+n}|z|^n \leq p^2 \tag{6.8}$$

with the aid of (4.14), (6.8) is true if

$$(p+n)|z|^n \leq \frac{p^2}{(p+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)} \tag{6.9}$$

Solving (6.9) for $|z|$, we get

$$|z| \leq \left\{\frac{p^2}{(p+1)(p+n)\, A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}\right\}^{1/n} \quad (n \geq 1) \tag{6.10}$$

Finally, since $(p+n)^{-1/n}$ is an increasing function for integers $n \geq 1,\ p \in \mathbb{N}$, we have (6.5) for $|z| < r_5$, where r_5 is given by (6.4).

In order to complete the proof of Theorem 6.1, we note that the result is sharp for the function $f(z) \in T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ of the form

$$f(z) = z^p - \frac{(p+1)A_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}{(p+n)} z^{p+n} \tag{6.11}$$

for some integers $n \geq 1$. □

Similarly we can prove the next theorem.

Theorem 6.2. *Let the function $f(z)$ defined by (1.2) be in the class $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. Then $f(z)$ is p-valent convex in the disk $|z| < r_6$, where*

$$r_6 = \inf_{n\in\mathbb{N}} \left\{\frac{p^2}{(p+1)(p+n)\, B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}\right\}^{1/n} \tag{6.12}$$

and $B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ is given by (4.24). The result is sharp for the function $f(z) \in C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ of the form

$$f(z) = z^p - \frac{(p+1)B_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)}{(p+n)} z^{p+n} \tag{6.13}$$

for some integers $n \geq 1$.

7. Conclusion

We have studied new classes $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ of p-valent functions with negative coefficients defined by a certain fractional derivative operator in the unit disk $\mathcal{U}$. Many of the known results follow as particular cases from our results; see for example, Aouf and Hossen (2006); Gupta (1984); Srivastava and Owa (1991a) and Srivastava and Owa(1991b). We obtained the sufficient conditions for the function $f(z)$ to be in $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. In addition, we derived a number of distortion theorems of functions belonging to these classes as well as distortion theorems for a certain fractional derivative operator of functions in the classes $T^*_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$ and $C_{\lambda,\mu,\eta}(p,\alpha,\beta,\gamma)$. At the end of this paper we have determined the radii of convexity for functions belonging to these classes

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