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A COEFFICIENT INEQUALITY FOR A SUB-CLASS OF ANALYTIC FUNCTIONS

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Abstract . In this paper a coefficient inequality for a subclass of analytic functions is obtained. Results proved by various authors will follow as special cases from our theorem.

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1. Introduction

Let U be the class of bounded functions

$$w(z) = \sum_{k=1}^{\infty} c_k z^k \tag{1.1}$$

which are regular in E and satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$.

It is well known [6] that $|c_1| \le 1$, $|c_2| \le 1 - |c_1|^2$.

Let A denote the class of functions

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ regular in } E. \tag{1.2}$$

Let $S^*(\beta)$ be the class of starlike functions of order β of the form (1.2) and with the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \text{, for some } 0 \le \beta < 1, \quad z \in E. \tag{1.3}$$

$K(\beta)$ is the class of convex functions of order β of the form (1.2) with the condition

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta\,,\, 0 \le \beta < 1\,,\, z \in E. \tag{1.4}$$

Obviously $S^*(0) \equiv S$ and $K(0) \equiv K$.

The classes $S^*(\beta)$ and $K(\beta)$ were introduced by Robertson[7].

A function $f(z) \in A$ is called α - convex function of order β ($0 \le \beta < 1$) if it satisfies the following conditions

(i) $\dfrac{f(z).f'(z)}{z} \ne 0\,,\quad z \in E.$

(ii) $\text{Re}\, J(\alpha; f) > \beta$ where

$$J(\alpha; f) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right), z \in E\,, \alpha \text{ any real number.} \tag{1.5}$$

The class of α -convex functions of order β is denoted by $M(\alpha, \beta)$.

Clearly $M(\alpha,0) \equiv M(\alpha)$, the class introduced by Mocanu[5].

It is known [4] that all α -convex functions are α -starlike in *E*.

A function $f(z)$ is said to be subordinate to function $g(z)$,if there exists a function $w(z) \in U$ such that $f(z) = g(w(z))$ and we write as $f(z) \prec g(z)$.

Let $M(\alpha; A, B; \delta)$ be the class of functions $f(z) \in A$ and with the conditions

(i) $\dfrac{f(z).f'(z)}{z} \ne 0$ and for $\alpha \ge 0$

(ii) $(1-\alpha)\dfrac{zf'(z)}{f(z)} + \alpha\left(1+\dfrac{zf''(z)}{f'(z)}\right) \prec \left(\dfrac{1+Az}{1+Bz}\right)^{\delta}$, $-1 \le B < A \le 1$, $0 < \delta \le 1$.

By A_p, we denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \;\; (p \text{ is a positive integer) regular in } E\,. \tag{1.6}$$

If $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ and $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$ are in A_p, then convolution or Hadamard product of $f(z)$ and $g(z)$ is denoted by $f * g$ and defined as

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}\;.$$

Let $T_{n,p}(\alpha; A, B; \delta)$ be the class of functions $f(z) \in A_p$ and satisfying the conditions

$$\frac{f(z).f'(z)}{z^{2p-1}} \ne 0 \qquad \text{and}$$

$$J_{n,p}(\alpha; f) = (1-\alpha)\frac{z\left(D_{n-1}^{p} f(z)\right)'}{D_{n-1}^{p} f(z)} + \alpha\frac{z\left(D_{n}^{p} f(z)\right)'}{D_{n}^{p} f(z)} \prec p\left(\frac{1+Az}{1+Bz}\right)^{\delta}, \qquad (1.7)$$

where n is any integer greater than $-p$, $-1 \le B < A \le 1$, $0 < \delta \le 1$, $\alpha \ge 0$,

$$D_{n-1}^{p} f(z) = \frac{z^{p}}{(1-z)^{n+p}} * f(z).$$

$*$ stands for convolution of functions.

By $S^{*}(A, B; \delta)$,we denote the class of functions $f(z) \in A$ regular in E with the condition

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}, -1 \le B < A \le 1, 0 < \delta \le 1. \qquad (1.8)$$

Obviously $S^{*}(A, B;1) \equiv S^{*}(A, B)$,which is a subclass of starlike functions studied by Goel and Mehrok [1] .

Let $K(A, B; \delta)$ be the class of functions $f(z) \in A$ and satisfying the condition

$$1 + \frac{zf''(z)}{f'(z)} \prec \left(\frac{1+Az}{1+Bz}\right)^{\delta}, -1 \le B < A \le 1, 0 < \delta \le 1 \qquad (1.9)$$

Clearly $K(A, B;1) \equiv K(A, B)$, which is a subclass of Convex functions studied by Goel and Mehrok [1].

We have the following observations:

(i) $T_{0,1}(\alpha; A, B; \delta) \equiv M(\alpha; A, B; \delta)$.

(ii) $T_{0,1}(\alpha; A, B;1) \equiv M(\alpha; A, B)$.

(iii) $T_{0,1}(0; A, B; \delta) \equiv S^{*}(A, B; \delta)$.

(iv) $T_{0,1}(0;1,-1;1) \equiv S^{*}$.

(v) $T_{0,1}(0;1-2\beta,-1;1) \equiv S^{*}(\beta)$.

(vi) $T_{0,1}(1; A, B; \delta) \equiv K(A, B; \delta)$.

(vii) $T_{0,1}(1;1,-1;1) \equiv K$.

In this paper, we obtain sharp bounds of $\left|a_{p+2} - \mu a_{p+1}^{2}\right|$ for the class $T_{n,p}(\alpha; A, B; \delta)$.

Results due to Keogh and Merkes [3], Szynal [8] and Goel and Mehrok [2] follows as special cases from our theorem.

2. Main Result

Theorem 2.1 If $f \in T_{n,p}(\alpha; A, B; \delta)$, then

(i) for μ complex ,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} \dfrac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}; |\lambda-\mu| \le \gamma \\ \dfrac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}|\lambda-\mu|; |\lambda-\mu| \ge \gamma \end{cases} \quad (2.1)$$

(ii) for μ real ,

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \le \begin{cases} \dfrac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}(\lambda-\mu); \mu \le \lambda-\gamma \\ \dfrac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}; \lambda-\gamma \le \mu \le \lambda+\gamma \\ \dfrac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}(\mu-\lambda); \mu \ge \lambda+\gamma \end{cases} \quad (2.2)$$

where

$$\lambda = \frac{p\delta(A-B)\left[(n+p)^2 + \alpha(2n+2p+1)\right] + (n+p+\alpha)^2\left[\dfrac{\delta(A-B)}{2} - \dfrac{(A+B)}{2}\right]}{p\delta(A-B)(n+p+1)(n+p+2\alpha)} \quad (2.3)$$

and $$\gamma = \frac{(n+p+\alpha)^2}{p\delta(A-B)(n+p+1)(n+p+2\alpha)}. \quad (2.4)$$

Proof. As $f \in T_{n,p}(\alpha; A, B; \delta)$, therefore

$$J_{n,p}(\alpha; f) = (1-\alpha)\frac{z\left(D_{n-1}^p f(z)\right)'}{D_{n-1}^p f(z)} + \alpha\frac{z\left(D_n^p f(z)\right)'}{D_n^p f(z)} \prec p\left(\frac{1+Az}{1+Bz}\right)^\delta .$$

By definition of subordination,

$$(1-\alpha)\frac{z\left(D_{n-1}^p f(z)\right)'}{D_{n-1}^p f(z)} + \alpha\frac{z\left(D_n^p f(z)\right)'}{D_n^p f(z)} = p\left(\frac{1+Aw(z)}{1+Bw(z)}\right)^\delta \text{,where } w(z) \in U . \quad (2.5)$$

An easy calculation shows that

$$\frac{z\left(D_{n-1}^{p}f(z)\right)'}{D_{n-1}^{p}f(z)}=p+(n+p)a_{p+1}z+(n+p)\left[(n+p+1)a_{p+2}-(n+p)a_{p+1}^{2}\right]z^{2}+\ldots.. \quad (2.6)$$

Replacing n by n+1 in (2.6) ,we have

$$\frac{z\left(D_{n}^{p}f(z)\right)'}{D_{n}^{p}f(z)}=p+(n+p+1)a_{p+1}z+(n+p+1)\left[(n+p+2)a_{p+2}-(n+p+1)a_{p+1}^{2}\right]z^{2}+\ldots.. \quad (2.7)$$

Using (2,6) and (2.7) , (2.5) becomes

$$p+(n+p+\alpha)a_{p+1}z+\left\{(n+p+1)(n+p+2\alpha)a_{p+2}-[(n+p)^{2}+\alpha(2n+2p+1)]a_{p+1}^{2}\right\}z^{2}+\ldots\ldots$$

$$=p+p\delta(A-B)c_{1}z+\left[p\delta(A-B)\left(c_{2}-Bc_{1}^{2}\right)+\frac{p\delta(\delta-1)}{2}(A-B)^{2}c_{1}^{2}\right]z^{2}+\ldots\ldots. \quad (2.8)$$

On equating coefficients in (2.8) , we get

$$a_{p+1}=\frac{p\delta(A-B)}{(n+p+\alpha)}c_{1} \quad (2.9)$$

and

$$a_{p+2}=\frac{p\delta(A-B)\left(c_{2}-Bc_{1}^{2}\right)+\frac{p\delta(\delta-1)}{2}(A-B)^{2}c_{1}^{2}+\left[(n+p)^{2}+\alpha(2n+2p+1)\right]\frac{p^{2}\delta^{2}(A-B)^{2}c_{1}^{2}}{(n+p+\alpha)^{2}}}{(n+p+1)(n+p+2\alpha)} \quad (2.10)$$

From (2.9) and (2.10) ,we have

$$a_{p+2}-\mu a_{p+1}^{2}=\frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}c_{2}$$

$$+\frac{p^{2}\delta^{2}(A-B)^{2}}{(n+p+\alpha)^{2}}\left\{\frac{p\delta(A-B)\left[(n+p)^{2}+\alpha(2n+2p+1)\right]+(n+p+\alpha)^{2}\left[\frac{\delta(A-B)}{2}-\frac{(A+B)}{2}\right]}{p\delta(A-B)(n+p+1)(n+p+2\alpha)}-\mu\right\}c_{1}^{2}.$$

$$\text{So}\left|a_{p+2}-\mu a_{p+1}^{2}\right|\leq\frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}\left|c_{2}\right|+\frac{p^{2}\delta^{2}(A-B)^{2}}{(n+p+\alpha)^{2}}|\lambda-\mu|\left|c_{1}^{2}\right|. \quad (2.11)$$

By using $|c_2| \le 1-|c_1|^2$, (2.11) becomes

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}+\frac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}\left[|\lambda-\mu|-\gamma\right]|c_1|^2. \qquad (2.12)$$

If $|\lambda-\mu| \le \gamma$, then from (2.12)

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}.$$

If $|\lambda-\mu| \ge \gamma$, then again from (2.12)

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}|\lambda-\mu|.$$

Now we discuss the case when μ is real .

Case I. for $\mu \le \lambda$, from (2.12)

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}+\frac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}\left[(\lambda-\gamma)-\mu\right]|c_1|^2. \qquad (2.13)$$

If $\mu \le \lambda-\gamma$, then (2.13) becomes

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}(\lambda-\mu).$$

If $\lambda-\gamma \le \mu \le \lambda$, from (2.13)

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}.$$

Case II. for $\mu \ge \lambda$, from (2.12)

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}+\frac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}\left[\mu-(\lambda+\gamma)\right]|c_1|^2. \qquad (2.14)$$

If $\mu \le \lambda+\gamma$, then from (2.14)

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p\delta(A-B)}{(n+p+1)(n+p+2\alpha)}.$$

If $\mu \ge \lambda+\gamma$, from (2.14)

$$\left|a_{p+2}-\mu a_{p+1}^2\right| \le \frac{p^2\delta^2(A-B)^2}{(n+p+\alpha)^2}(\mu-\lambda).$$

On putting $n=0$, $p=1$, $\alpha=0$ in the above theorem , we have the following:

Cor 2.1 If $f \in S^{*}(A,B;\delta)$, then

(i) for μ complex ,

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{\delta(A-B)}{2}; \left|\lambda_1-\mu\right| \le \gamma_1 \\ \delta^2(A-B)^2\left|\lambda_1-\mu\right|; \left|\lambda_1-\mu\right| \ge \gamma_1 \end{cases}$$

(ii) for μ real ,

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \delta^2(A-B)^2(\lambda_1-\mu); \mu \le \lambda_1-\gamma_1 \\ \dfrac{\delta(A-B)}{2}; \lambda_1-\gamma_1 \le \mu \le \lambda_1+\gamma_1 \\ \delta^2(A-B)^2(\mu-\lambda_1); \mu \ge \lambda_1+\gamma_1 \end{cases}$$

where $\lambda_1 = \dfrac{(A-B)(3\delta-1)-2B}{4\delta(A-B)}$ and $\gamma_1 = \dfrac{1}{2\delta(A-B)}$.

If we put $n=0$, $p=1$, $\alpha=1$ in theorem (2.1) , we get the following result:

Cor 2.2 If $f \in K(A,B;\delta)$ **,** then

(i) for μ complex ,

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{\delta(A-B)}{6}; \left|\lambda_2-\mu\right| \le \gamma_2 \\ \dfrac{\delta^2(A-B)^2}{4}\left|\lambda_2-\mu\right|; \left|\lambda_2-\mu\right| \ge \gamma_2 \end{cases}$$

(ii) for μ real ,

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{\delta^2(A-B)^2}{4}(\lambda_2-\mu); \mu \le \lambda_2-\gamma_2 \\ \dfrac{\delta(A-B)}{6}; \lambda_2-\gamma_2 \le \mu \le \lambda_2+\gamma_2 \\ \dfrac{\delta^2(A-B)^2}{4}(\mu-\lambda_2); \mu \ge \lambda_2+\gamma_2 \end{cases}$$

where $\lambda_2 = \dfrac{(A-B)(3\delta-1)-2B}{3\delta(A-B)}$ and $\gamma_2 = \dfrac{2}{3\delta(A-B)}$.

By putting $n=0$, $p=1$, $\alpha=0$, $A=1$, $B=-1$, $\delta=1$ in theorem (2.1), we have the following result due to Keogh and Merkes [3].

Cor 2.3 If $f\in S^*$, then

$$\left|a_3-\mu a_2^2\right|\le \max\{1,|4\mu-3|\}.$$

For $n=0$, $p=1$, $\alpha=1$, $A=1$, $B=-1$, $\delta=1$, the following result is obvious:

Cor 2.4 If $f\in K$, then

$\left|a_3-\mu a_2^2\right|\le \max\left\{\frac{1}{3},|1-\mu|\right\}$, which is also a result due to Keogh and Merkes [3].

If we take $n=0$, $p=1$, $A=1-2\beta$, $B=-1$, $\delta=1$ in the theorem (2.1), we obtain the following result due to Szynal [8].

Cor 2.5 For $f\in S^*(\beta)$, we have

$$\left|a_3-\mu a_2^2\right|\le \frac{1-\beta}{1+2\alpha}\max\left\{1,\frac{|4\mu(1-\beta)(1+2\alpha)+4(1-\beta)-(1+\alpha)(7+\alpha-6\beta)|}{(1+\alpha)^2}\right\}.$$

If we put $n=0$, $p=1$ in theorem (2.1), we get the following result:

Cor 2.6 If $f\in M(\alpha;A,B;\delta)$, then

(i) for μ complex,

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{\delta(A-B)}{2(1+2\alpha)}; & |\lambda_3-\mu|\le\gamma_3\\[2ex] \dfrac{\delta^2(A-B)^2}{(1+\alpha)^2}|\lambda_3-\mu|; & |\lambda_3-\mu|\ge\gamma_3\end{cases}$$

(ii) for μ real,

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{\delta^2(A-B)^2}{(1+\alpha)^2}(\lambda_3-\mu); & \mu\le\lambda_3-\gamma_3\\ \dfrac{\delta(A-B)}{2(1+2\alpha)}; & \lambda_3-\gamma_3\le\mu\le\lambda_3+\gamma_3\\ \dfrac{\delta^2(A-B)^2}{(1+\alpha)^2}(\mu-\lambda_3); & \mu\ge\lambda_3+\gamma_3\end{cases}$$

where $\lambda_3=\dfrac{\delta(A-B)[1+3\alpha]+(1+\alpha)^2\left[\dfrac{\delta(A-B)}{2}-\dfrac{(A+B)}{2}\right]}{2\delta(A-B)(1+2\alpha)}$ and $\gamma_3=\dfrac{(1+\alpha)^2}{2\delta(A-B)(1+2\alpha)}$.

Putting $n=0$, $p=1$, $\delta=1$ in theorem (2.1) , we have

Cor 2.7 If $f\in M(\alpha;A,B)$, then

(i) for μ complex ,

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{(A-B)}{2(1+2\alpha)}; & |\lambda_4-\mu|\le\gamma_4\\ \dfrac{(A-B)^2}{(1+\alpha)^2}|\lambda_4-\mu|; & |\lambda_4-\mu|\ge\gamma_4\end{cases}$$

(ii) for μ real ,

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{(A-B)^2}{(1+\alpha)^2}(\lambda_4-\mu); & \mu\le\lambda_4-\gamma_4\\ \dfrac{(A-B)}{2(1+2\alpha)}; & \lambda_4-\gamma_4\le\mu\le\lambda_4+\gamma_4\\ \dfrac{(A-B)^2}{(1+\alpha)^2}(\mu-\lambda_4); & \mu\ge\lambda_4+\gamma_4\end{cases}$$

where $\lambda_4=\dfrac{(A-B)[1+3\alpha]-B(1+\alpha)^2}{2(A-B)(1+2\alpha)}$ and $\gamma_4=\dfrac{(1+\alpha)^2}{2(A-B)(1+2\alpha)}$.

These results were due to Goel and Mehrok [2].

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