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A MIXED QUADRATURE RULE FOR NUMERICAL INTEGRATION OF ANALYTIC FUNCTIONS BY USING BIRKHOFF YOUNG AND BOOLS QUADRATURE

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A quadrature rule has been constructed for the approximate evaluation of the integral of an analytic function over a line segment in complex plane. The rule has been numerically verified and its asymptotic error estimate has been obtained.

Key words: Quadrature rule, Asymptotic error, Analytic function, Numerical integration.

1. Introduction

There are several rules for the approximate evaluation of real definite integral.

$$\int_{-1}^{1} f(z)dz \qquad ....(1.1)$$

However there are only a few quadrature rules for evaluating an integral of type

$$I(f) = \int_{L} f(z)dz \qquad ....(1.2)$$

Where L is a directed line segment from the point Z_0-h to Z_0+h in the complex plane and $f(z)$ is analytic in certain domain $\boldsymbol{\Omega}$ containing L.

Birkhoff - Young (1950) [3] have derived the following interpolatory type of quadrature rule:

$$BY(f) = (h/15)[24\, f(Z_0) + 4\,\{ f(Z_0 + h) + f(Z_0 - h)\} - \{ f(Z_0 + ih) + f(Z_0 - ih)\}] \quad \text{........(1.3)}$$

and this rule is of precision five.

Lether (1976) using the transformation $Z = Z_0 + ht$, t [-1,1] transformed the integral (1.2) to the integral

$$h\int_{-1}^{1} f\,(Z_0 + ht)\,dt\,. \quad \text{....... (1.4)}$$

and then made the approximation of this integral by applying standard quadrature rule meant for approximate evaluation of real definite integral (1.1). The rule so formed is termed as a TRANSFORMED RULE for the numerical integration of (1.2).

R.N. Das and G.Pradhan (1996) [1] have constructed a quadrature rule for the approximate evaluation of the integral (1.1) from two standard quadrature rules of different type but of equal precision, such rule is termed as MIXED QUADRATURE RULE.

In this paper we desire to construct a mixed quadrature rule of precision seven in the same vein for the approximation of the integral (1.2).

2. Formulation of the rule

For the construction of the desired rule we choose the rule (3) and the following Bools quadrature transformed rule

$$BL(f) = (h/45)\,[7\,f(Z_0 - h) + 7\,f(Z_0 + h) + 32\,f(Z_0 - h/2) + 32\,f(Z_0 + h/2) + 12\,f(Z_0)] \quad \text{........(2.1)}$$

Each of the rules under consideration is of precision five.

Denoting the truncation errors by E_{BY} and E_{BL} due to rules (1.3) and (2.1) respectively in approximating the integral (1.2) , we have

$$I(f) = BY(f) + E_{BY} \quad \text{............(2.2)}$$

and

$$I(f) = BL(f) + E_{BL} \quad \text{...........(2.3)}$$

Where f is infinitely differentiable since it is assumed to be analytic in certain domain containing the line segment L. So by using Taylor's expansion the truncation errors associated with the quadrature rules under reference can be expressed as

$$E_{By}(f) = (-1/1890)h^7 f^{vi}(Z_0) - (1/226800)h^9 f^{viii}(Z_0) \quad \ldots\ldots\ldots\ldots(2.4)$$

and

$$E_{BL}(f) = (-1/15120)\, h^7 f^{vi}(Z_0) - (17/7257600)\, h^9 f^{viii}(Z_0) \quad \ldots\ldots\ldots(2.5)$$

Now multiplying (2.2) and (2.3) by 1/15120 and 1/1890 respectively and subtracting the 1st one from the second we obtained

$$I(f) = [(8/7)\ BL(f) - (1/7)\ BY(f)] + [(8/7)\ E_{BL} - (1/7)\ E_{BY}]$$

$$= SM_2(f) + E_{SM2}$$

where $SM_2(f) = [(8/7)\ BL(f) - (1/7)\ BY(f)]$ ....(2.6)

is desired quadrature rule of precision seven for the approximate evaluation of $I(f)$ and the truncation error committed in this approximation is given by

$$E_{SM2} = [(8/7)\, E_{BL}(f) - (1/7)\ E_{BY}(f)] \quad \ldots\ldots\ldots\ldots(2.7)$$

The rule (2.6) may be called a MIXED TYPE QUADRATURE RULE as it is constructed from two different type of rules of same precision.

3. Error Analysis

Let $f(Z)$ is analytic in the disc $\mathbf{\Omega}_R = \{ Z : |z - Z_0| \leq R > |h| \}$

So that the points Z_0, $Z_0 \pm h$, $Z_0 \pm ih$ are all interior to the disc $\mathbf{\Omega}_R$. Now using Taylor's expansion

$$f(Z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad ; \quad a_n = \left(\frac{1}{n!}\right) f^{n}(Z_0)$$

in (2.7) we obtain after simplification

$$E_{SM2} = (h^9/226800)\ f^{viii}(Z_0) + \ldots \quad \ldots\ldots\ldots\ldots(3.1)$$

Thus the mixed rule SM_2 is of precision seven.

From (3.1) we have the following theorem.

Theorem 3.1 If f is assumed to be analytic in a domain Ω containing L then

$$E_{SM2}(f) = 0(h^9)$$

Error Comparison

From (2.4) and (2.5) we get

$$|E_{BL}| \leq |E_{BY}| \qquad \text{..........(3.2)}$$

Again from (3.1)

$$|E_{SM2}| \leq |E_{BL}| \qquad \text{..........(3.3)}$$

From (3.2) and (3.3) $|E_{SM2}| \leq |E_{BL}| \leq |E_{BY}|$

4. Numerical Verification

Let us approximate the value of the following integral using BY(f), BL (f) and $SM_2(f)$ quadrature rule

Consider $I_1 = \int_{-i}^{i} e^z dz$ and $I_2 = \int_{-i/2}^{i/2} \cos z dz$

Table for comparison of values of integrals I_1 and I_2

Sl.No	Quadrature Rules	Approximation(I_2)	Approximation(I_1)
1	BY (f)	i (1.6824171)	i(1.0421948)
2	BL (f)	i (1.6828781)	i(1.0421911)
3	SM_2 (f)	i (1.682944)	i(1.0421906)
4	Exact Value	i (1.6829420)	i(1.0421906)

5. Conclusion

From the table it is evident that the mixed quadrature rule $SM_2(f)$ is giving us better result than each of the BL(f) (Bool's rule) and BY(f) (Birkhoff-Young's rule).This conforms the theoretical result that $SM_2(f)$ is of higher precision (precision 7) than those of BY(f) and BL(f) (each having precision 5).

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