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NUMBERS : RATIONAL, IRRATIONAL OR TRANSCENDENTAL ?

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We talk of rational numbers, irrational numbers, algebraic numbers, transcendental numbers and briefly describe some real numbers π, Euler's number e, Euler's Constant γ, Liouville constant α. The first two numbers π and Euler's number e are used extensively in undergraduate courses, in fact π is known to all of us from schooldays. We then mention some real numbers for which it is still not known whether they are rational, irrational or transcendental?

Keywords: Rational number, transcendental number, Liouville number.

1. Introduction

The number system of mathematics begins with the counting numbers $1, 2, 3, \ldots$ which are called natural numbers and is denoted by N. If $m, n \in N$ then we cannot always solve the equation $x + m = n$ in N and so from N we arrive at the set of integers Z in which we can solve the above type of equations. Again if $a, b \in Z$ and $a \neq 0$ then we cannot always solve the equation $ax = b$ in Z. So from Z we move to the set of rational numbers Q by taking the solutions of the above equations. Every non-empty bounded above subset of Q does not have a least upper bound in Q. We move from Q to R to ensure that every non-empty bounded above subset of R has a least upper bound in R. Moving from Q to the set of real numbers R is not as simple as moving from N to Z or from Z to Q. It requires the concept of Dedekind cut or Cauchy sequence of rationals or sequence of nested intervals. Our aim is not to talk of construction of real numbers from rational numbers in which one can define a real number as an element of a complete ordered field. One can say

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that N is incomplete w.r.t +, Z is incomplete w.r.t ., Q is incomplete w.r.t. $<$ and R is complete w.r.t all of them. We here talk of the real numbers which are termed as rational, irrational, algebraic, transcendental, Liouville numbers in the set of real numbers and try to look at the nature of some real numbers like Euler's number e, π and Euler's Constant γ. For the sake of clarity we first give the following definitions and then raise some pertinent questions and try to answer them.

Definition 1.1. ***Rational number***
A real number x is said to be a rational number iff it can be written in the form $x = \frac{a}{b}$ where a,b are integers and $b \neq 0$. One can interpret a rational number as follows:
The ratio of two given real numbers a and b ($b \neq 0$) is said to be a rational number or the two numbers a and b is said to be commensurable iff there exists some integer k such that when a is divided into k equal parts it turns out that the second number b can also be divided into m such equal parts for some integer m. This interpretation becomes more clear for a positive rational number.
With this interpretation we can say that $x = \frac{2\sqrt{3}\pi}{5\sqrt{3}\pi}$ is a rational number, here a is divided into 2 equal parts each of length $l = \sqrt{3}\pi$ and b is divided into 5 such equal parts.

Definition 1.2. ***Irrational number***
A real number x is said to be an irrational number iff it is not a rational number.

Definition 1.3. ***Algebraic number***
A real number a is said to be an algebraic number if the number a satisfies an equation of the form

$$c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = 0$$

where the coefficients $c_n, c_{n-1}, \ldots c_0$ are all rational numbers and $c_n \neq 0$.
One can assume that the coefficients are all integers for, if $c_n = \frac{p_n}{q_n},\ c_{n-1} = \frac{p_{n-1}}{q_{n-1}}, \ldots c_0 = \frac{p_0}{q_0}$ and $d_n = p_n q_{n-1} q_{n-2} \ldots q_1 q_0,\ d_{n-1} = q_n p_{n-1} q_{n-2} \ldots q_1 q_0,\ d_0 = q_n q_{n-1} \ldots q_1 p_0$. Then a satisfies an equation of the form

$$d_n x^n + d_{n-1} x^{n-1} + \ldots + d_1 x + d_0 = 0$$

where the coefficients $d_n, d_{n-1}, \ldots d_0$ are all integers and $d_n \neq 0$.
Thus a real number a is an algebraic number iff a satisfies a polynomial equation with rational or integral coefficients.

Definition 1.4. ***Transcendental number***
A real number x is said to be a transcendental number iff it is not an algebraic number.

Once we described the rational , irrational, algebraic and transcendental numbers we now raise the following questions and try to answer them :

Q.1. Whether $\sqrt{p}$ ***is rational for a prime p and more generally whether*** $p^{\frac{1}{n}}$ ***is rational for*** $n > 1$ ***?***

Q.2. What is the relation between rational numbers and algebraic numbers?

Q.3. Are the following numbers rational : Euler's number e, the number π***, Euler's constant*** γ ***? Are the numbers transcendental?***

Q.4. What is the nature of the number $10^{-1!} + 10^{-2!} + 10^{-3!} + \dots$ ***? Is it irrational? Is it transcendental?***

Q.5. What about the nature of the numbers of the form

$$2^{\sqrt{2}},\ 3^{\sqrt{5}},\ e^e, \alpha^\beta$$

where α ***and*** β ***are not algebraic ?***

The answer to the first question follow from the theorem given below:

Theorem 1.5. *If the polynomial equation* $d_n x^n + d_{n-1}x^{n-1} + \ldots + d_1 x + d_0 = 0$ *with integral coefficients has a rational root* $\frac{a}{b}$ *then* $a|d_0$ *and* $b|d_n$.

Proof. Suppose the polynomial equation $d_n x^n + d_{n-1}x^{n-1} + \ldots + d_1 x + d_0 = 0$ with integral coefficients has a rational root $\frac{a}{b}$. Then we get

$$\begin{aligned} d_n \frac{a^n}{b^n} + d_{n-1}\frac{a^{n-1}}{b^{n-1}} + \ldots\ldots + d_1 \frac{a}{b} + d_0 = 0 \\ \Rightarrow d_n a^n + b d_{n-1} a^{n-1} + \ldots + b^{n-1} d_1 a + b^n d_0 = 0 \\ \Rightarrow d_n a^n + b d_{n-1} a^{n-1} + \ldots\ldots + b^{n-1} d_1 a = -b^n d_0 \\ \Rightarrow b^n d_0 + b^{n-1} d_1 a + \ldots\ldots + b d_{n-1} a^{n-1} = -d_n a^n \end{aligned}$$

From the last two equations we get for some integers k_1 and k_2

$$a k_1 = -b^n d_0 \quad \textit{and} \quad b k_2 = -d_n a^n.$$

As a and b has no common factor this shows that $a|d_0$ and $b|d_n$.

Corollary 1.6. *If the polynomial equation* $x^n + d_{n-1}x^{n-1} + \ldots + d_1 x + d_0 = 0$ *with integral coefficients has a rational root then it must be an integer which divides* d_0.

Proof. Let the rational root be $\frac{a}{b}$. Then $a|d_0$ and $b|1$ so that $b = 1$. Thus the rational root is an integer which divides d_0.

As an application of this corollary we see that the equation $x^n - p = 0$ has no rational root for $n > 1$ and p a prime. So $p^{\frac{1}{n}}$ is not a rational number. Further argument wil ensure that for $a \in N$ $a^{\frac{1}{n}}$ is a rational number if a is of the form b^n

for some $b \in N$. Thus $\sqrt{2}, \sqrt{5}, 7^{\frac{2}{3}}$ are all irrational numbers.

From the definition of an algebraic number it follows that every rational number $\frac{a}{b}$ is an algebraic number as the number $\frac{a}{b}$ satisfies the equation $bx - a = 0$. The converse is however not true as for a given prime p, $p^{\frac{1}{n}}$ is an algebraic number, for it satisfies the equation $x^n - p = 0$ but it is not a rational number. **The observations made here answer Q. No. 2.**

We next try to answer Q.No.3

Euler's Number e : The number e was first studied by the Swiss mathematician Euler in 1720s, although its existence was more or less implied in the work of John Napier, the inventor of logarithms in 1614. Euler was the first to use the letter e for it in 1727 (the fact that it is the first letter of his surname is coincidental). As a result e is called the Euler's number or Napier's constant. e is defined as the limit of the convergent sequence $(1 + \frac{1}{n})^n$ or as the infinite sum $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots$. Euler gave an approximation of e upto 18 decimal places as e = 2.718281828459045235. Now its value has been calculated to 869,894,101 decimal places. We give a proof that e is irrational.
If possible let $e = \frac{h}{k}$ where h and k are integers with $k > 0$. Then clearly $k!.(e - \frac{1}{0!} - \frac{1}{1!} - \frac{1}{2!} - \ldots - \frac{1}{k!})$ is an integer but

$$\begin{aligned}
0 &< k!(e - \frac{1}{0!} - \frac{1}{1!} - \frac{1}{2!} - \ldots - \frac{1}{k!}) \\
&= \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} + \ldots \\
&< \frac{1}{k+1} + \frac{1}{(k+1)^2} + \frac{1}{(k+1)^3} + \ldots \\
&= \frac{1}{(k+1)} . \frac{1}{1 - \frac{1}{k+1}} = \frac{1}{k} \\
&\leq 1
\end{aligned}$$

Thus $0 < k!(e - \frac{1}{0!} - \frac{1}{1!} - \frac{1}{2!} - \ldots - \frac{1}{k!}) < 1$. This is a contradiction and so e must be irrational □
The transcendence of e was first proved by Hermite[Hermite (1873)]. A much easier proof of transcendence of e is given in the book by Niven [Niven (2005)] which is based on a paper by Hurwitz [Hurwitz (1893)]. Anyone interested to know further about e can look at the book " e: the story of a number " by Eli Maor [Maor (2009)].

The number π : π is defined as the ratio of a circle to its diameter. Archimedes (287 B.C. - 212 B.C.) was the first to give a method of calculating π to any desired degree of accuracy by the method of exhaustion. The irrationality of π was first proved by J.H. Lambert [Lambert (1761)]. We give the proof by Niven [Niven (1947)]
.

If possible let π be rational. Then π^2 is also rational. Let $\pi^2 = \frac{a}{b}$ where a and b are positive integers. Consider

$$f : [0,1] \longrightarrow R \ \ defined \ \ by \ \ f(x) = \frac{x^n(1-x)^n}{n!} \ \ where \ \ n \in N.$$

Clearly $0 < f(x) < \frac{1}{n!}$ for all x in (0,1). Also k-th derivative $f^{(k)}(0)$ and $f^{(k)}(1)$ are all integers. Let us define another function $F : [0,1] \longrightarrow R$ by

$$F(x) = b^n \sum_{k=0}^{n} (-1)^k \pi^{2n-2k} f^{(2k)}(x).$$

Then clearly F(0) and F(1) are integers. Also

$$\frac{d}{dx}\{F'(x)\sin \pi x - \pi F(x)\cos \pi x\} = F^{(2)}(x)\sin \pi x + \pi^2 F(x)\sin \pi x = \pi^2 a^n f(x) \sin \pi x$$

$$\Rightarrow \int_0^1 \pi^2 a^n f(x)\sin \pi x dx = [F'(x)\sin \pi x - \pi F(x)\cos \pi x]_{x=0}^1 = \pi[F(1) + F(0)]$$

$$\Rightarrow \int_0^1 \pi a^n f(x)\sin \pi x dx = F(1) + F(0) - \ \ an \ \ integer$$

Now $0 < f(x) < \frac{1}{n!}$ $\forall n \in N \ and \ for \ 0 < x < 1$ and so $0 < \pi a^n \int_0^n f(x)\sin \pi x dx < \frac{\pi a^n}{n!}$ $\forall n \in N$.
By choosing n sufficiently large we can make $\frac{\pi a^n}{n!} < 1$ and so we get a contradiction. Thus π^2 must be irrational and so also is π $\square$
The number π is not algebraic. Its transcedency was first proved by Lindemann [Lindemann (1882)] . A nice history of π can be found in the book “ A History of π (pi) ” by Petr Beckmann [Beckman (1976)] and in the book by Blatner [Blatner (1999)].

Euler's Constant γ : Euler's constant γ is defined as

$$\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - logn).$$

The existence of the limit follows from the **Integral test**:
Let f be a positive decreasing function on $[1,\infty)$ *such that* $\lim_{x\to\infty} f(x) = 0$. *For* $n = 1, 2, \ldots$ *define* $s_n = \sum_{k=1}^n f(k), t_n = \int_1^n f(x)dx, d_n = s_n - t_n$. *Then we have*
(i) $0 < f(n+1) \le d_{n+1} \le d_n \le f(1), n = 1, 2, \ldots$
(ii) $\lim_{n\to\infty} d_n$ *exists*
(iii) $\sum_{n=1}^\infty f(n)$ *converges iff* $\{t_n\}$ *converges*
(iv) $0 \le d_k - \lim_{n\to\infty} d_n \le f(k)$ *for* $k = 1, 2, \ldots$
Choosing $f(x) = \frac{1}{x}$ the existence of γ is justified.
Although γ is calculated upto 1,000,000 digits it is still unknown whether γ is rational or not. γ was calculated to 16 digits by Euler in 1781. Its value is .57721566...... . If γ is a simple fraction $\frac{a}{b}$ then it is proved that $b > 10^{1242080}$. As we are not certain about the rationality of γ so we can't decide the transcendency of γ. For further information on γ one can look at the book “ Gamma : Exploring Euler's

constant " by J. Havil [Havil (2009)].

Next we discuss problem 4. In 1851, the French Mathematician Liouville [Liouville (1844)]first established that transcendental numbers exist by exhibiting certain numbers which he proved to be non-algebraic. These numbers are now called Liouville numbers. A real number ξ is said to be a Liouville number if for every positive integer m there is a rational number $\frac{h_m}{k_m}$ with $k_m > 1$ such that $|\xi - \frac{h_m}{k_m}| < {k_m}^{-m}$. The number α is one such number and is called the Liouville's constant.

To answer Q.No.5 we start with Hilbert's 7th problem. In 1900 Hilbert announced a list of 23 outstanding problems. The seventh problem was to decide whether α^β is algebraic or not, given that α and β are algebraic numbers. In 1934 it was settled by Gelfond [Gel'fond (1934)] and independently by Schneider [Schneider (1934)] that α^β is transcendental (the case $\alpha = 0,\ \alpha = 1$ and $\beta = \ rational$ were excluded). So $2^{\sqrt{2}},\ 3^{\sqrt{5}}$ are transcendental . The number e^π is transcendental but it is still not known whether $e^e,\ \pi^e$ is algebraic or not. The problem like whether α^β is transcendental for α, β not algebraic are still open.
Even it is not known whether $\pi + e$, $\pi - e$, $\pi.e$, π/e are transcendental or not ?

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