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BOUNDED INVERSE RELATIONAL FUNCTION

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In the present paper,we find the relationship between three functions, Bijective function, Inverse of the function and Identity function and their behaviour in the bounded region.

Keywords: Mean Value Theorem, Bounded Inverse Theorem, continuity, Differentiability, Monotonic function, Graphs.

1. Introduction

An Identity function is a function that always returns the same value that was used as its argument. In terms of equation, the function is given by $f(x) = x$. A Bijective function is a function f from set X to set Y with the property that for every y in Y, there is exactly one x in X, such that $f(x) = y$ and no unmapped element remains in both X and Y. Bijective function plays a fundamental role in many areas of mathematics, for instance in the definition of Isomorphism, Permutation group, Projective map and many others. Let f be a function whose domain is the set X, and whose co-domain is the set Y. Then if it exists, the inverse of f is the function f^{-1} with domain Y and co-domain X defined by: $f(x) = y$ and $f^{-1}(y) = x$ i.e a function is invertible if and only if its inverse relation is a function in which case the inverse relation is the inverse function. The inverse relation is the relation obtained by switching x and y every where.In this paper we study the relationship between Bijective function, Inverse of the function and Identity function, in a interval. Generally theorems are derived on individual function, while in this paper we derive results on the combination of functions.

2. Theorem.

Let there be any function say f : [a,b] → [d,e],
Then if :
(1) The function f(x) is continuous in [a,b]
(2) The function f(x) is differentiable in (a,b)

(3) The function f(x) is Bijective in [a,b]
(4) The sign of both a , b and f(a) , f(b) are opposite
(5) Either : $|f(a)| < |a|$ and $|f(b) < |b|$ or $|f(a)| > |a|$ and $|f(b)| > |b|$

Let $f^{-1}(x) = g(x)$
Then there exists at least one point c such that :

$$f(c) = g(c) = c \quad and \tag{1}$$

$$f'(c) \times g'(c) = 1 \;\; for \;\; f'(c) = g'(c) \neq 0 \tag{2}$$

where c is the point of intersection of function $f(x), g(x)$ and $y = x$,
and g : [d,e] → [a,b]

$$For \qquad f(c) = g(c) = c \tag{3}$$

2.1. *Proof.1:*

2.1.1. *Case : (1)*

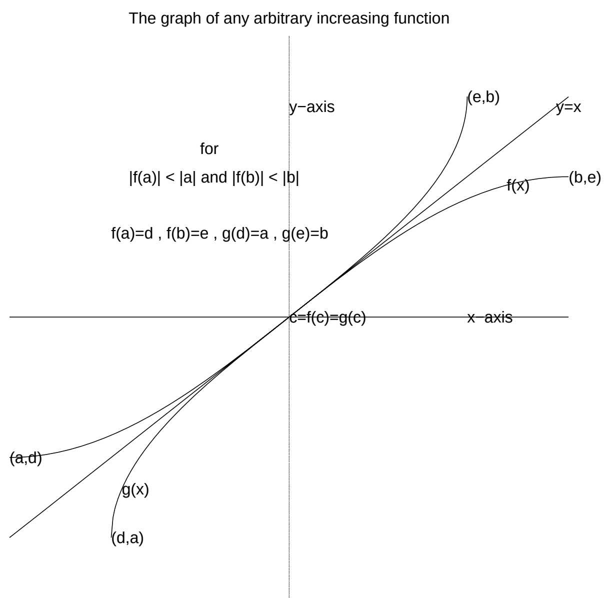
Let the function f(x) is strictly increasing for $a < b$ and $f(a) < f(b)$.Since a and b have opposite signs and f(a) and f(b) also have opposite signs. Then the co-ordinate [a,f(a)] and [b,f(b)] will lie in the opposite quadrant. When $|f(a)| < |a|$ and $|f(b)| < |b|$ then the point (a,f(a)) will lie above line y=x and the point (b,f(b)) will below line y=x. Since the co-ordinates (a,f(a)) and (b,f(b)) lie on the opposite side of the line y=x. The function f(x) is continuous between point (a,f(a)) and (b,f(b)). So there must exist atleast a single point say (k,f(k)) at the intersection of curve f(x) and line y=x. Hence k=f(k).

2.1.2. *Case : (2)*

Let the function f(x) is strictly increasing then function g(x) will also be strictly increasing. since $g(d) < g(e)$ for $d < e$ then the co-ordinates [d,g(d)] and [e,g(e)] will lie in the opposite quadrant. when $|f(a)| < |a|$ and $|f(b)| < |b|$ then $|g(d)| > |d|$ and $|g(e)| > |e|$ and the points (d,g(d)) will lie below line y=x and point (e,g(e)) will above line y=x. Since the co-ordinate (d,g(d)) and (e,g(e)) lie on the opposite side of the line y=x. The function g(x) is continuous between point (d,g(d)) and (e,g(e)). So there must exist at least a single point say (l,g(l)) at the intersection of curve g(x) and line y=x. Hence l=g(l).

2.1.3. *Case : (3)*

Let the function f(x) and its inverse g(x) are the reflection image about the line y=x and the co-ordinates of x and y are interchange by the property of inverse of the function. Let (m,f(m)) be any arbitrary point nearest to the line y=x on the curve of the function f(x) then its corresponding nearest point on the curve of the function g(x) will be (f(m),m). If the curve of the function f(x) intersect the line y=x at any



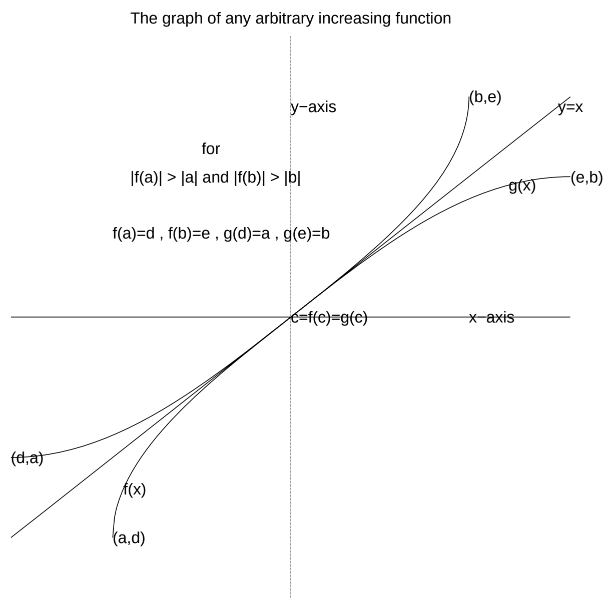
point then the inverse g(x) of the function will also intersect at the same point. So m=f(m)=g(m). From cases (1),(2) and(3) we get k=f(k)=g(k), l=f(l)=g(l) and m=f(m)=g(m). Hence f(c)=g(c)=c.

2.1.4. *Case : (4)*

Let function f(x) is strictly increasing for $a < b$, $f(a) < f(b)$. Let a and b have opposite sign and f(a) and f(b) also have opposite sign. Then the co-ordinate [a,f(a)] and [b,f(b)] will lie in the opposite quadrant. When $|f(a)| > |a|$ and $|f(b)| > |b|$ then the point (a,f(a)) will lie below line y=x and the point (b,f(b)) be will above line y=x. The co-ordinate (a,f(a)) and (b,f(b)) lie on the opposite side of the line y=x. Since the function f(x) is continuous between point (a,f(a)) and (b,f(b)). So there must exist at least a single point say (n,f(n)) at the intersection of curve f(x) and line y=x. Hence n=f(n).

2.1.5. *Case : (5)*

Let the function f(x) is strictly increasing then function g(x) will also be strictly increasing and $g(d) < g(e)$ for $d < e$. Let d and e have opposite sign and g(d) and g(e) also have opposite sign. Then the co-ordinate [d,g(d)] and [e,g(e)] will lie in the opposite quadrant. When $|f(a)| > |a|$ and $|f(b)| > |b|$ then $|g(d)| < |d|$ and $|g(e)| < |e|$. Then the point (d,g(d)) will lie above line y=x and the point (e,g(e)) will be below the line y=x. The co-ordinate (d,g(d)) and (e,g(e)) lie on the opposite side of the line y=x and function g(x) is continuous between point



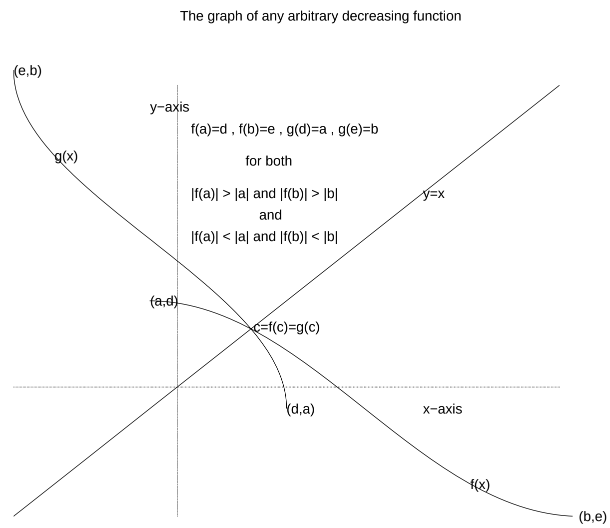
(d,g(d)) and (e,g(e)). Then there must exist at least a single point say (o,g(o)) at the intersection of curve g(x) and line y=x. So, o=g(o). From cases (3),(4) and (5) we get n=f(n)=g(n), o=f(o)=g(o) and m=f(m)=g(m). Hence f(c)=g(c)=c.

2.1.6. *Case : (6)*

Let function f(x) is strictly decreasing for $a < b$ and $f(a) > f(b)$. Let a and b have opposite sign and f(a) and f(b) also have opposite sign. Then the co-ordinate [a,f(a)] and [b,f(b)] will lie in the opposite quadrant. If $a < b$ and $f(a) > f(b)$ then a and b have opposite sign and a and f(a) will also have opposite sign. since $a < 0$ then $f(a) > 0$ and $b > 0$ then $f(b) < 0$. Since a is negative and f(a) is positive so co-ordinate [a,f(a)] will always lie above the line y=x also b is positive and f(b) is negative. Hence [b,f(b)] will always lie below the line y=x. Since the co-ordinates (a,f(a)) and (b,f(b)) lie on the opposite side of the line y=x. The function f(x) is continuous between the points (a,f(a)) and (b,f(b)). So there must exist at least one single point say (p,f(p)) at the intersection of curve f(x) and line y=x. Hence p=f(p).

2.1.7. *Case : (7)*

Let function f(x) is strictly decreasing then its inverse g(x) is also be strictly decreasing $g(e) > g(d)$ for $e < d$. If d and e have opposite sign and g(d) and g(e) also have opposite sign. Then the co-ordinates [d,g(d)] and [e,g(e)] will lie in the opposite quadrant. Since both $e < d$ and $g(e) > g(d)$ and a and b have opposite



sign for $e < 0$ then $g(e) > 0$ and $d > 0$ then $g(d) < 0$. Since e is negative and g(e) is positive so co-ordinate [e,g(e)] will always lie above the line y=x also d is positive and g(d) is negative hence [d,g(d)] will always lie below the line y=x. The co-ordinate (d,g(d)) and (e,g(e)) lie on the opposite side of the line y=x and the function g(x) is continuous between points (d,g(d)) and (e,f(e)). Hence there must exist at least a single point say (q,g(q)) at the intersection of curve g(x) and line y=x. So q=g(q). From cases (3),(6) and (7) we get p=f(p)=g(p), q=f(q)=g(q) and m=f(m)=g(m). Hence f(c)=g(c)=c

$$For \qquad f'(c) \times g'(c) = 1 \tag{4}$$

2.2. *Proof 2:*

Since both the functions f(x) and g(x) are continuous and differentiable and they intersect each other at any point on line y=x say [c,f(c)] or [c,g(c)] since f(c)=g(c)=c Now the equation of the tangent on the curve of function f(x) will be:

$$f'(c) = \frac{y - c}{x - c} \tag{5}$$

And the equation of the tangent on the curve of function g(x) will be:

$$g'(c) = \frac{y - c}{x - c} \tag{6}$$

On solving equation (5) and equation (6), we get:

$$y - x \times f'(c) + c[f'(c) - 1] = 0 \tag{7}$$

and

$$y - x \times g'(c) + c[g'(c) - 1] = 0 \tag{8}$$

Then the equation of bisectors of both the tangents is given by:

$$\frac{y - x \times f'(c) + c[f'(c) - 1]}{\sqrt{1 + [f'(c)]^2}} = + \frac{y - x \times g'(c) + c[g'(c) - 1]}{\sqrt{1 + [g'(c)]^2}} \tag{9}$$

and

$$\frac{y - x \times f'(c) + c[f'(c) - 1]}{\sqrt{1 + [f'(c)]^2}} = -\frac{y - x \times g'(c) + c[g'(c) - 1]}{\sqrt{1 + [g'(c)]^2}} \tag{10}$$

On differentiating equation (9) and equation (10) with respect to x , we get:

$$\frac{y' - f'(c)}{\sqrt{1 + [f'(c)]^2}} = +\frac{y' - g'(c)}{\sqrt{1 + [g'(c)]^2}} \tag{11}$$

$$and \quad \frac{y' - f'(c)}{\sqrt{1 + [f'(c)]^2}} = -\frac{y' - g'(c)}{\sqrt{1 + [g'(c)]^2}} \tag{12}$$

2.2.1. *Case : (1)*

Since the graph of f(x) and g(x) are reflection image about line y=x. Hence one of the angle bisectors of both the tangents gives the equation of line y=x. Since the slope of line y=x is given by $y' = 1$. Substituting $y' = 1$ into (11) and (12) we get:

$$\frac{1 - f'(c)}{\sqrt{1 + [f'(c)]^2}} = +\frac{1 - g'(c)}{\sqrt{1 + [g'(c)]^2}} \tag{13}$$

$$and \quad \frac{1 - f'(c)}{\sqrt{1 + [f'(c)]^2}} = -\frac{1 - g'(c)}{\sqrt{1 + [g'(c)]^2}} \tag{14}$$

On squaring both sides of (13) and (14) we get:

$$\frac{[1 - f'(c)]^2}{1 + [f'(c)]^2} = \frac{[1 - g'(c)]^2}{1 + [g'(c)]^2} \tag{15}$$

On solving and simplifying equation (15), we get:

$$[1 + (f'(c))^2 - 2f'(c)] \times [1 + (g'(c))^2] = [1 + (g'(c))^2 - 2g'(c)] \times [1 + (f'(c))^2] \tag{16}$$

$$1 + (f'(c))^2 - 2f'(c) + (g'(c))^2 + [f'(c) \times g'(c)]^2 - \tag{17}$$

$$2f'(c) \times (g'(c))^2 = 1 + (g'(c))^2 - 2g'(c) + \tag{18}$$

$$(f'(c))^2 + [g'(c) \times f'(c)]^2 - 2g'(c) \times (f'(c))^2 \tag{19}$$

$$f'(c) + f'(c) \times (g'(c))^2 = g'(c) + g'(c) \times (f'(c))^2 \tag{20}$$

$$f'(c) \times g'(c)[f'(c) - g'(c)] - [f'(c) - g'(c)] = 0 \tag{21}$$

$$(f'(c) \times g'(c) - 1) \times (f'(c) - g'(c)) = 0 \tag{22}$$

$$hence \;\; f'(c) \times g'(c) = 1 \;\; for \;\; f'(c) = g'(c) \neq 0 \tag{23}$$

2.2.2. *Case : (2)*

Since one of the angle bisectors of both the tangents gives the equation of line y=x. So the other angle bisector of the tangents will be perpendicular to line y=x at that point. Since product of slope of two lines perpendicular to each other is equal to -1. So the slope of second bisector is given by $y' = -1$.

Substituting $y' = -1$ into (11) and (12) we get:

$$\frac{-1 - f'(c)}{\sqrt{1 + [f'(c)]^2}} = +\frac{-1 - g'(c)}{\sqrt{1 + [g'(c)]^2}} \tag{24}$$

$$and \ \ \frac{-1 - f'(c)}{\sqrt{1 + [f'(c)]^2}} = -\frac{-1 - g'(c)}{\sqrt{1 + [g'(c)]^2}} \tag{25}$$

On squaring both sides (24) and (25) we get:

$$\frac{[1 + f'(c)]^2}{1 + [f'(c)]^2} = \frac{[1 + g'(c)]^2}{1 + [g'(c)]^2} \tag{26}$$

On solving and simplifying equation (26) we get:

$$[1 + (f'(c))^2 + 2f'(c)] \times [1 + (g'(c))^2] = [1 + (g'(c))^2 + 2g'(c)] \times [1 + (f'(c))^2] \tag{27}$$

$$1 + (f'(c))^2 + 2f'(c) + (g'(c))^2 + [f'(c) \times g'(c)]^2 + \tag{28}$$

$$2f'(c) \times (g'(c))^2 = 1 + (g'(c))^2 + 2g'(c) + \tag{29}$$

$$(f'(c))^2 + [g'(c) \times f'(c)]^2 + 2g'(c) \times (f'(c))^2 \tag{30}$$

$$f'(c) + f'(c) \times (g'(c))^2 = g'(c) + g'(c) \times (f'(c))^2 \tag{31}$$

$$f'(c) \times g'(c)[f'(c) - g'(c)] - [f'(c) - g'(c)] = 0 \tag{32}$$

$$(f'(c) \times g'(c) - 1) \times (f'(c) - g'(c)) = 0 \tag{33}$$

$$hence \ \ f'(c) \times g'(c) = 1 \ \ for \ \ f'(c) = g'(c) \neq 0 \tag{34}$$

We conclude from case(1) and (2) that $f'(c) \times g'(c) = 1$ for all cases except $f'(c) = g'(c) \neq 0$.

2.3. *Conclusion*

We have studied the relationship between Bijective function, Inverse of the function and Identity function, in a definite interval. We have derived some standard results which may be useful in all branch of Science. Three particles (heavenly bodies, vehicles, satellites) are moving in a plane with their respective path equations $y = x$, $y = f(x)$, and $y = f^{-1}(x)$. If they satisfy the above stated conditions, then they can meet at a common point and the products of their velocities will be equal to one. This theorem is first of its kind which combines three function and gives standard results.

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