THE SUM OF ONE SERIES
Euler and Bernoulli summation methods of inverted squares series

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The series of inverted squares and its sum can be investigated in several ways. We offer two of them - the Euler method and Bernoulli method using product of series.

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The series \( \sum \frac{1}{n^2} \) is convergent and its sum is

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

We usually prove it using e.g. Fourier series and the Parseval equality. The purpose of this text is to recollect another possibilities how to calculate this sum. These methods are interesting and maybe more suitable for students.

The first one (Euler method) does not present strict proof. It concerns relations between roots and coefficients of finite polynomials and generalizes these relations for "infinite" polynomials.

The second one (Bernoulli method) uses product of series (not absolutely convergent).

1. Euler Method

Leonhard Euler generalized properties of polynomials to infinite series.

At first we consider a quadratic equation

\[
0 = a_2 x^2 + a_1 x + a_0 = a_2 \cdot (x - x_1) \cdot (x - x_2)
\]

and we find out relations between its coefficients \( a_1, a_2 \) and its solutions \( x_1, x_2 \)

\[
\frac{a_0}{a_2} = x_1 \cdot x_2, \quad \frac{-a_1}{a_2} = x_1 + x_2, \quad \text{hence} \quad \frac{a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2}.
\]

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The same is true for a cubic equation

\[ 0 = a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0 = a_3 \cdot (x - x_1) \cdot (x - x_2) \cdot (x - x_3). \]

We have again

\[ \frac{-a_0}{a_3} = x_1 \cdot x_2 \cdot x_3, \quad \frac{a_1}{a_3} = x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3, \quad \text{and} \quad \frac{-a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}. \]

The similar relation holds for equations of fourth order and so on. In general

\[ \frac{-a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \quad (1) \]

for

\[ 0 = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0 = a_n \cdot (x - x_1) \cdot (x - x_2) \cdots \cdots (x - x_n). \]

Now we apply analogically this equality to an equation of "infinite" order. Using expansion of a function \( \sin \) into power series

\[
\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots, \quad \text{or}
\]

\[
\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \ldots
\]

and substitution \( t := \sqrt{x} \) we can rewrite equation \( \frac{\sin \sqrt{x}}{\sqrt{x}} = 0 \) into

\[
0 = \frac{\sin \sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{1440} + \ldots .
\]

Let us remind that \( a_0 = 1 \) and \( a_1 = -\frac{1}{6} \). Solutions of such equation are \( x_k = k^2 \cdot \pi^2 \), where \( k \in \mathbb{N} \), as \( x > 0 \) and \( \sqrt{x} = k \cdot \pi \).

Hence we can analogically conclude according to (1) for "infinite" polynomials

\[
\frac{1}{6} = -\frac{a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \ldots \quad \text{and}
\]

\[
\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots .
\]

2. Bernoulli Method

Another method is based on products of sums

\[
1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan 1 = \frac{\pi}{4} = s
\]

(2)

which is convergent by Leibnitz test. The equality (2) can be derived from sum of geometrical series

\[
1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2} \quad \text{by integration} \quad x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \arctan x .
\]
The main idea of the following method goes back to Nicolaus Bernoulli. Product of sequences \((a_0, -a_1, a_2, -a_3, \ldots) = (1, -\frac{1}{3}, \frac{1}{5}, -\frac{1}{7}, \ldots)\) gives us something like infinite matrix or infinite table.

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
\frac{1}{3} & \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \frac{1}{13} \\
\frac{1}{9} & \frac{1}{15} & \frac{1}{21} & \frac{1}{27} & \frac{1}{33} & \frac{1}{39} \\
\frac{1}{15} & \frac{1}{25} & \frac{1}{35} & \frac{1}{45} & \frac{1}{55} & \frac{1}{65} \\
\frac{1}{7} & \frac{1}{21} & \frac{1}{35} & \frac{1}{49} & \frac{1}{63} & \frac{1}{77} \\
\frac{1}{9} & \frac{1}{27} & \frac{1}{45} & \frac{1}{63} & \frac{1}{81} & \frac{1}{99} \\
\frac{1}{11} & \frac{1}{33} & \frac{1}{55} & \frac{1}{77} & \frac{1}{99} & \frac{1}{121} \\
\frac{1}{13} & \frac{1}{39} & \frac{1}{65} & \frac{1}{91} & \frac{1}{117} & \frac{1}{143} \\
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

We shall denote by \(A_n, n = 0, 1, 2, \ldots\) the sum of diagonals and \(B_n, n = 1, 2, \ldots\) sum of skew diagonals according to the following schemas:

Fig. 1. The sums of diagonals and skew diagonals.

We want to investigate the sum \(A_0\). It holds \(2 \cdot A_n = B_n\) for \(n = 1, 2, \ldots\). We can add that infinite matrix by squares (partial sums \(Q_n\)), then we can add the diagonals (partial sums \(S_n\)) and the skew diagonals (partial sums \(T_n\)).
All three methods of summation give us the same number, which equals to 

\[ s^2 = \left( \frac{\pi}{4} \right)^2, \]

hence from

\[ S_n = A_0 - 2A_1 + 2A_2 - 2A_3 + \cdot + (-1)^n \ldots 2A_n = \]

\[ = A_0 - B_1 + B_2 - B_3 + \cdot + (-1)^n \ldots B_n = A_0 - T_n \]

we have

\[ s^2 = A_0 - s^2 \quad \text{and so} \quad A_0 = 2s^2 = \frac{\pi^2}{8}. \]

Now more precisely and correctly. The series (2) is not absolutely convergent. That is why we cannot use Mertens’ theorem and we must use other ways for proof.

It is clear that the series

\[ A_0 = a_0^2 + a_1^2 + a_2^2 + a_3^2 + \cdots = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots, \]

is convergent by the integral test or Cauchy condensation test for instance. The partial sums by squares are convergent and

\[ Q_n = (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1})^2 \to s^2 = \frac{\pi^2}{16}. \]

2.1. Relations between the sums \( A_n \) and \( B_n \) and convergence of them

At first we shall consider sums according to the diagonals \( A_n \).

We can use the decomposition of fraction

\[ \frac{2}{3} = 1 - \frac{1}{3}, \quad \frac{2}{15} = \frac{1}{3} - \frac{1}{5}, \quad \frac{4}{5} = 1 - \frac{1}{5}, \quad \frac{4}{21} = \frac{1}{3} - \frac{1}{7}, \ldots \]

for the series \( A_1, A_2 \) etc.
\[ A_1 = \frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \cdots = \frac{1}{2} \cdot \left( 1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} + \cdots \right) = \frac{1}{2} \cdot 1 , \]

\[ A_2 = \frac{1}{7} + \frac{1}{35} + \frac{1}{135} + \cdots = \frac{1}{4} \cdot \left( 1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} + \cdots \right) = \frac{1}{4} \cdot \left( 1 + \frac{1}{4} \right) , \]

\[ A_3 = \frac{1}{9} + \frac{1}{63} + \frac{1}{315} + \cdots = \frac{1}{6} \cdot \left( 1 - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} - \frac{1}{9} + \cdots \right) = \frac{1}{6} \cdot \left( 1 + \frac{1}{3} + \frac{1}{6} \right) , \]

and so on. In general we have

\[ A_n = \frac{1}{2n} \cdot (1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}) . \]

There are also interesting relations between sums \( B_n \) and \( A_n \)

\[ 2A_1 = 1 = B_1 , \]
\[ 4A_2 = 1 + \frac{1}{4} = \frac{2}{3} = \frac{2 \cdot (\frac{1}{3} + \frac{1}{4})}{2} = 2B_2 , \]
\[ 6A_3 = 1 + \frac{1}{3} + \frac{1}{5} = (1 + \frac{1}{3}) + \frac{1}{5} = 3 \cdot (\frac{2}{3} + \frac{1}{9}) = 3B_3 , \]
\[ 8A_4 = 1 + \frac{1}{4} + \frac{1}{6} + \frac{1}{7} = (1 + \frac{1}{3}) + (\frac{3}{4} + \frac{1}{7}) = 4 \cdot (\frac{7}{15} + \frac{1}{15} + \frac{1}{4} + \frac{1}{9}) = 4B_4 , \]
\[ 10A_5 = 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} = (1 + \frac{1}{3}) + (\frac{1}{2} + \frac{1}{7}) + \frac{1}{8} = 5 \cdot (\frac{1}{2} + \frac{1}{21} + \frac{1}{25}) = 5B_5 , \]

and so on, in general

\[ 2A_n = B_n . \]

The sequence \( A_n \) is decreasing and \( \lim A_n = 0 \) because of well known inequality

\[ \limsup \frac{a_1 + a_2 + \cdots + a_n}{n} \leq \limsup a_n . \]

So the series \( \sum \left( -1 \right)^{n+1}A_n \) is convergent by Leibniz test and we can denote

\[ \sum_{n=1}^{\infty} \left( -1 \right)^{n+1}A_n = A \quad \text{or} \quad \lim S_n = A \quad . \]

As \( B_n = 2A_n \) holds, the same is true for sum \( \sum \left( -1 \right)^{n+1}B_n \) and

\[ \sum_{n=1}^{\infty} \left( -1 \right)^{n+1}B_n = 2A \quad \text{or} \quad \lim T_n = 2A \quad . \]
2.2. Sums $Q_n$ and $S_n$

Now we shall consider the sums $Q_n$ and $S_n$. It holds

$$S_n = A_0 - 2A_1 + 2A_2 - 2A_3 + \cdots + 2 \cdot (-1)^n A_n - A_0 - 2A.$$ 

Now we shall focus our interest on the difference of $S_n$ and $T_n$ and we define sequences $A_{nk}$ and $C_n$

$$-(S_1 - Q_1) = 2 \left( A_1 - \frac{1}{3} \right) - \left( \frac{1}{25} + \frac{1}{49} + \ldots \right) \quad \text{where}$$

$$A_{11} = \frac{1}{15} + \frac{1}{33} + \frac{1}{69} + \cdots = \frac{2}{15} + \frac{2}{33} + \frac{2}{69} + \ldots = \frac{2}{15} \cdot \frac{2}{3},$$

$$S_2 - Q_2 = 2 \left( A_2 - \frac{1}{3} \right) - \left( A_1 - \frac{1}{3} - \frac{1}{15} \right) + \frac{1}{39} + \frac{1}{81} + \ldots \quad \text{where}$$

$$A_{22} = \frac{1}{27} + \frac{1}{65} + \frac{1}{7} + \cdots = \frac{2}{27} + \frac{2}{65} + \frac{2}{77} + \cdots = \frac{2}{27} \cdot \frac{2}{7},$$

$$A_{21} = \frac{1}{35} + \frac{1}{63} + \frac{1}{79} + \cdots = \frac{4}{35} + \frac{4}{63} + \frac{4}{77} + \cdots = \frac{4}{35} \cdot \frac{4}{7}.$$

$$-(S_3 - Q_3) = 2 \left( A_3 - \frac{1}{5} \right) - 2 \left( A_2 - \frac{1}{5} - \frac{1}{21} \right) + 2 \left( A_1 - \frac{1}{5} - \frac{1}{15} - \frac{1}{35} \right) -$$

$$\cdots \quad \text{where}$$

$$A_{33} = \frac{1}{27} + \frac{1}{55} + \frac{1}{91} + \cdots = \frac{6}{27} + \frac{6}{57} + \frac{6}{91} + \ldots = \frac{6}{27} \cdot \frac{6}{91},$$

$$A_{32} = \frac{1}{45} + \frac{1}{77} + \frac{1}{117} + \cdots = \frac{4}{45} + \frac{4}{77} + \frac{4}{117} + \ldots = \frac{4}{45} \cdot \frac{4}{77},$$

$$A_{31} = \frac{1}{63} + \frac{1}{99} + \frac{1}{143} + \cdots = \frac{2}{63} + \frac{2}{99} + \frac{2}{143} + \cdots = \frac{2}{63} \cdot \frac{2}{143},$$

and so on. In general

$$(-1)^n \cdot (S_n - Q_n) = 2 \cdot A_{nn} - 2 \cdot A_{n-1} + \cdots + (-1)^n - 1 \cdot 2 \cdot A_{n1} + (-1)^n \cdot C_{n+1}$$

where $A_{nm} = \frac{1}{2m} \left( \frac{2n + 1}{2n + 1} + \frac{1}{2n - 1} + \cdots + \frac{1}{2(n - m) + 3} \right),$ 

$$C_n = \frac{1}{(2n + 1)^2} + \frac{1}{(2n + 3)^2} + \ldots \quad \text{for} \quad 1 \leq m \leq n.$$
Simple calculation give us the following inequalities and limits

\[ A_{n1} \leq A_{n2} \leq \cdots \leq A_{n,n-1} \leq A_{nn} \quad \text{and} \quad 0 \leq A_{nn} - A_{n,n-1} + \cdots + (-1)^{n-1} \cdot A_{n1} \leq A_{nn} \leq A_n \rightarrow 0 \]

\[ C_n \rightarrow 0 \, . \]

From

\[ |S_n - Q_n| \leq 2 \cdot |A_{nn} - A_{n,n-1} + \cdots + (-1)^{n-1} \cdot A_{n1}| + |C_{n+1}| \leq [A_{nn} \leq A_n \rightarrow 0] \]

we have

\[ S_n - Q_n \rightarrow 0 \]

and hence the limit of both partial sums must approach the same value and

\[ A_0 - 2A = s^2 \, . \]

2.3. Sums \( T_n \) and \( Q_n \)

Inequalities between sums \( T_n \) and \( Q_n \)

\[ T_1 = B_1 - B_2 \leq Q_1 \leq B_1 - B_2 + B_3 = T_2 \, , \]

\[ T_3 = B_1 - B_2 + B_3 - B_4 \leq Q_3 \leq B_1 - B_2 + B_3 - B_4 + B_5 - B_6 + B_7 = T_6 \, , \]

\[ T_5 = B_1 - B_2 + B_3 - B_4 + B_5 - B_6 \leq Q_5 \leq B_1 - B_2 + B_3 - B_4 + B_5 - B_6 + B_7 - B_8 + B_9 - B_{10} + B_{11} = T_{10} \, , \]

generally

\[ T_n = B_1 - B_2 + \cdots + (-1)^n \cdot B_{n+1} \leq Q_n \leq B_1 - B_2 + \cdots + B_{2n+1} = T_{2n} \]

for any odd \( n \) give us

\[ \lim T_n \leq \lim Q_n \leq \lim T_n, \quad 2A \leq s^2 \leq 2A \quad \text{and} \quad 2A = s^2 \, . \] (4)

2.4. Conclusion

As we remind (3) and (4) we have

\[ s^2 = A_0 - 2A = A_0 - s^2, \quad \text{hence} \quad A_0 = 2s^2 = \frac{\pi^2}{8} \]

and

\[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = A_0 = \frac{\pi^2}{8} \, . \]
The investigated series can be divided into odd and even terms

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

and we can conclude

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.
\]

References
