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THE SUM OF ONE SERIES
Euler and Bernoulli summation methods of inverted squares series

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The series of inverted squares and its sum can be investigated in several ways. We offer two of them - the Euler method and Bernoulli method using product of series.

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The series $\sum \frac{1}{n^2}$ is convergent and its sum is

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ .$$

We usually prove it using e.g. Fourier series and the Parseval equality. The purpose of this text is to recollect another possibilities how to calculate this sum. These methods are interesting and maybe more suitable for students.

The first one (Euler method) does not present strict proof. It concerns relations between roots and coefficients of finite polynomials and generalizes these relations for "infinite" polynomials.

The second one (Bernoulli method) uses product of series (not absolutely convergent).

1. Euler Method

Leonhard Euler generalized properties of polymonials to infinite series.

At first we consider a quadratic equation

$$0 = a_2x^2 + a_1x + a_0 = a_2 \cdot (x - x_1) \cdot (x - x_2)$$

and we find out relations between its coefficients a_1, a_2 and its solutions x_1, x_2

$$\frac{a_0}{a_2} = x_1 \cdot x_2, \quad -\frac{a_1}{a_2} = x_1 + x_2 \ , \quad \text{hence} \quad -\frac{a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2} \ .$$

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The same is true for a cubic equation

$$0 = a_3 \cdot x^3 + a_2 \cdot x^2 + a_1 \cdot x + a_0 = a_3 \cdot (x - x_1) \cdot (x - x_2) \cdot (x - x_3) \ .$$

We have again

$$-\frac{a_0}{a_3} = x_1 \cdot x_2 \cdot x_3, \quad \frac{a_1}{a_3} = x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3, \quad \text{and} \quad -\frac{a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \ .$$

The similar relation holds for equations of fourth order and so on. In general

$$-\frac{a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \tag{1}$$

for

$$0 = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \cdots + a_1 \cdot x + a_0 = a_n \cdot (x - x_1) \cdot (x - x_2) \cdot \cdots \cdot (x - x_n) \ .$$

Now we apply analogically this equality to an equation of "infinite" order. Using expansion of a function sin into power series

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots \ , \ \text{or}$$
$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \ldots$$

and substitution $t := \sqrt{x}$ we can rewrite equation $\frac{\sin\sqrt{x}}{\sqrt{x}} = 0$ into

$$0 = \frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{6} + \frac{x^2}{120} - \frac{x^3}{1440} + \ldots \ .$$

Let us remind that $a_0 = 1$ and $a_1 = -\frac{1}{6}$. Solutions of such equation are $x_k = k^2 \cdot \pi^2$, where $k \in \mathbb{N}$, as $x > 0$ and $\sqrt{x} = k \cdot \pi$.

Hence we can analogically conclude according to (1) for "infinite" polynomials

$$\frac{1}{6} = -\frac{a_1}{a_0} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \frac{1}{16\pi^2} + \ldots \quad \text{and}$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots \quad .$$

2. Bernoulli Method

Another method is based on products of sums

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \arctan 1 = \frac{\pi}{4} = s \tag{2}$$

which is convergent by Leibniz test. The equality (2) can be derived from sum of geometrical series

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2} \ \text{ by integration } \ x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \arctan x \ .$$

The main idea of the following method goes back to Nicolaus Bernoulli. Product of sequences $(a_0, -a_1, a_2, -a_3, \dots) = (1,\ -\frac{1}{3},\ \frac{1}{5},\ -\frac{1}{7}, \dots)$ gives us something like infinite matrix or infinite table

$$\begin{array}{cccccccc}
1 & -\dfrac{1}{3} & \dfrac{1}{5} & -\dfrac{1}{7} & \dfrac{1}{9} & -\dfrac{1}{11} & \dfrac{1}{13} & \cdots \\
-\dfrac{1}{3} & \dfrac{1}{9} & -\dfrac{1}{15} & \dfrac{1}{21} & -\dfrac{1}{27} & \dfrac{1}{33} & -\dfrac{1}{39} & \cdots \\
\dfrac{1}{5} & -\dfrac{1}{15} & \dfrac{1}{25} & -\dfrac{1}{35} & \dfrac{1}{45} & -\dfrac{1}{55} & \dfrac{1}{65} & \cdots \\
-\dfrac{1}{7} & \dfrac{1}{21} & -\dfrac{1}{35} & \dfrac{1}{49} & -\dfrac{1}{63} & \dfrac{1}{77} & -\dfrac{1}{91} & \cdots \\
\dfrac{1}{9} & -\dfrac{1}{27} & \dfrac{1}{45} & -\dfrac{1}{63} & \dfrac{1}{81} & -\dfrac{1}{99} & \dfrac{1}{117} & \cdots \\
-\dfrac{1}{11} & \dfrac{1}{33} & -\dfrac{1}{55} & \dfrac{1}{77} & -\dfrac{1}{99} & \dfrac{1}{121} & -\dfrac{1}{143} & \cdots \\
\dfrac{1}{13} & -\dfrac{1}{39} & \dfrac{1}{65} & -\dfrac{1}{91} & \dfrac{1}{117} & -\dfrac{1}{143} & \dfrac{1}{169} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}.$$

We shall denote by A_n, $n = 0, 1, 2, \dots$ the sum of diagonals and B_n, $n = 1, 2, \dots$ sum of skew diagonals according to the following schemas

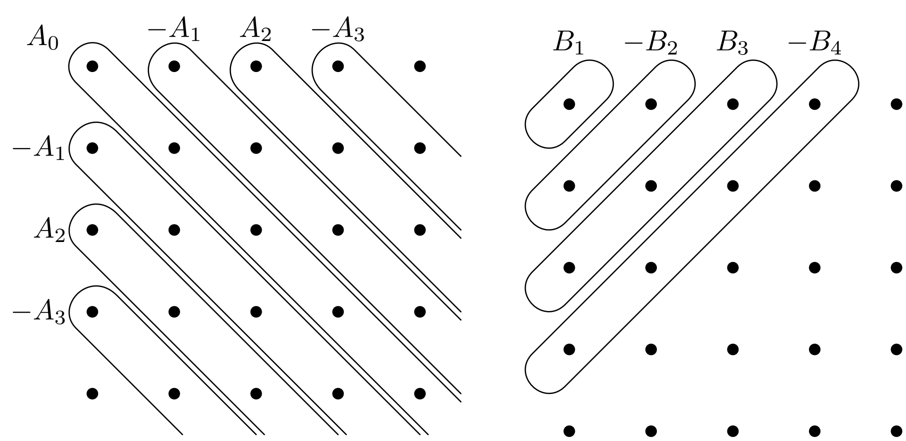

Fig. 1. The sums of diagonals and skew diagonals.

We want to investigate the sum A_0. It holds $2 \cdot A_n = B_n$ for $n = 1, 2, \dots$. We can add that infinite matrix by squares (partial sums Q_n), then we can add the diagonals (partial sums S_n) and the skew diagonals (partial sums T_n).

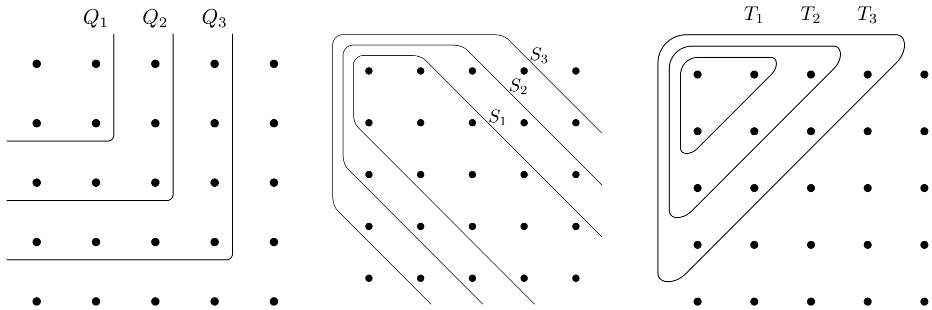

Fig. 2. The possibilities of summation.

All three methods of summation give us the same number, which equals to $s^2 = (\frac{\pi}{4})^2$, hence from

$$S_n = A_0 - 2A_1 + 2A_2 - 2A_3 + \cdot + (-1)^n \dots 2A_n =$$
$$= A_0 - B_1 + B_2 - B_3 + \cdot + (-1)^n \dots B_n = A_0 - T_n$$

we have

$$s^2 = A_0 - s^2 \quad \text{and so} \quad A_0 = 2s^2 = \frac{\pi^2}{8} \ .$$

Now more precisely and correctly. The series (2) is not absolutely convergent. That is why we cannot use Mertens' theorem and we must use other ways for proof.

It is clear that the series

$$A_0 = a_0^2 + a_1^2 + a_2^2 + a_3^2 + \cdots = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \dots \ ,$$

is convergent by the integral test or Cauchy condensation test for instance. The partial sums by squares are convergent and

$$Q_n = (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^n \frac{1}{2n+1})^2 \to s^2 = \frac{\pi^2}{16} \ .$$

2.1. *Relations between the sums A_n and B_n and convergence of them*

At first we shall consider sums according to the diagonals A_n.

We can use the decomposition of fraction

$$\tfrac{2}{3} = 1 - \tfrac{1}{3}\,, \ \tfrac{2}{15} = \tfrac{1}{3} - \tfrac{1}{5}\,, \ \tfrac{4}{5} = 1 - \tfrac{1}{5}\,, \ \tfrac{4}{21} = \tfrac{1}{3} - \tfrac{1}{7}\,, \dots$$

for the series A_1, A_2 etc.

$$
\begin{aligned}
A_1 &= \tfrac{1}{3} + \tfrac{1}{15} + \tfrac{1}{35} + \dots = \tfrac{1}{2} \cdot (\underbrace{1 - \cancel{\tfrac{1}{3}}}_{=\frac{2}{3}} + \underbrace{\cancel{\tfrac{1}{3}} - \cancel{\tfrac{1}{5}}}_{=\frac{2}{15}} + \underbrace{\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{7}}}_{=\frac{2}{35}} + \dots) = \tfrac{1}{2} \cdot 1 \ , \\
A_2 &= \tfrac{1}{5} + \tfrac{1}{21} + \tfrac{1}{45} + \tfrac{1}{77} + \dots = \\
&= \tfrac{1}{4} \cdot (\underbrace{1 - \cancel{\tfrac{1}{5}}}_{=\frac{4}{5}} + \underbrace{\tfrac{1}{3} - \cancel{\tfrac{1}{7}}}_{=\frac{4}{21}} + \underbrace{\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{9}}}_{=\frac{4}{45}} + \underbrace{\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{11}}}_{=\frac{4}{77}} + \dots) = \tfrac{1}{4} \cdot (1 + \tfrac{1}{3}) \ , \\
A_3 &= \tfrac{1}{7} + \tfrac{1}{27} + \tfrac{1}{55} + \tfrac{1}{91} + \dots = \\
&= \tfrac{1}{6} \cdot (\underbrace{1 - \cancel{\tfrac{1}{7}}}_{=\frac{6}{7}} + \underbrace{\tfrac{1}{3} - \cancel{\tfrac{1}{9}}}_{=\frac{6}{27}} + \underbrace{\tfrac{1}{5} - \cancel{\tfrac{1}{11}}}_{=\frac{6}{55}} + \underbrace{\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{13}}}_{=\frac{4}{91}} + \dots) = \tfrac{1}{6} \cdot (1 + \tfrac{1}{3} + \tfrac{1}{5})
\end{aligned}
$$

and so on. In general we have

$$A_n = \frac{1}{2n} \cdot (1 + \frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2n-1}) \ .$$

There are also interesting relations between sums B_n and A_n

$$
\begin{aligned}
2A_1 &= 1 = B_1 \ , \\
4A_2 &= 1 + \tfrac{1}{3} = \tfrac{2}{3} + \tfrac{2}{3} = 2 \cdot (\tfrac{1}{3} + \tfrac{1}{3}) = 2B_2 \ , \\
6A_3 &= 1 + \tfrac{1}{3} + \tfrac{1}{5} = (1 + \tfrac{1}{5}) + \tfrac{1}{3} = 3 \cdot (\tfrac{2}{5} + \tfrac{1}{9}) = 3 \cdot (\tfrac{1}{5} + \tfrac{1}{9} + \tfrac{1}{5}) = 3B_3 \ , \\
8A_4 &= 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} = (1 + \tfrac{1}{7}) + (\tfrac{1}{3} + \tfrac{1}{5}) = \\
&= 4 \cdot (\tfrac{2}{7} + \tfrac{2}{15}) = 4 \cdot (\tfrac{1}{15} + \tfrac{1}{7} + \tfrac{1}{7} + \tfrac{1}{15}) = 4B_4 \ , \\
10A_5 &= 1 + \tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7} + \tfrac{1}{9} = (1 + \tfrac{1}{9}) + (\tfrac{1}{3} + \tfrac{1}{7}) + \tfrac{1}{5} = \\
&= 5 \cdot (\tfrac{2}{9} + \tfrac{2}{21} + \tfrac{1}{25}) = 5 \cdot (\tfrac{1}{9} + \tfrac{1}{21} + \tfrac{1}{25} + \tfrac{1}{21} + \tfrac{1}{9}) = 5B_5 \ ,
\end{aligned}
$$

and so on, in general

$$2A_n = B_n \ .$$

The sequence A_n is decreasing and $\lim A_n = 0$ because of well known inequality

$$\limsup \frac{a_1 + a_2 + \dots + a_n}{n} \leq \limsup a_n \ .$$

So the series $\sum (-1)^{n+1} A_n$ is convergent by Leibniz test and we can denote

$$\sum_{n=1}^{\infty} (-1)^{n+1} A_n = A \quad (\text{or} \quad \lim S_n = A \ ) \ .$$

As $B_n = 2A_n$ holds, the same is true for sum $\sum (-1)^{n+1} B_n$ and

$$\sum_{n=1}^{\infty} (-1)^{n+1} B_n = 2A \quad (\text{or} \quad \lim T_n = 2A) \ .$$

2.2. *Sums Q_n and S_n*

Now we shall consider the sums Q_n and S_n. It holds

$$S_n = A_0 - 2A_1 + 2A_2 - 2A_3 + \dots + 2 \cdot (-1)^n A_n \to A_0 - 2A \ .$$

Now we shall focus our interest on the difference of S_n and T_n and we define sequences A_{nk} and C_n

$$-(S_1 - Q_1) = 2\underbrace{(A_1 - \tfrac{1}{3})}_{=A_{11}} - \underbrace{(\tfrac{1}{25} + \tfrac{1}{49} + \dots)}_{=C_2} \quad \text{where}$$

$$A_{11} = \tfrac{1}{15} + \tfrac{1}{35} + \tfrac{1}{63} + \dots = \tfrac{1}{2}(\underbrace{\tfrac{1}{3} - \cancel{\tfrac{1}{5}}}_{=\frac{2}{15}} + \underbrace{\cancel{\tfrac{1}{5}} - \cancel{\tfrac{1}{7}}}_{=\frac{2}{35}} + \underbrace{\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{9}}}_{=\frac{2}{63}} + \dots) = \tfrac{1}{2} \cdot \tfrac{1}{3} \ ,$$

$$S_2 - Q_2 = 2\underbrace{(A_2 - \tfrac{1}{5})}_{=A_{22}} - \underbrace{(A_1 - \tfrac{1}{3} - \tfrac{1}{15})}_{=A_{21}} + \underbrace{\tfrac{1}{49} + \tfrac{1}{81} + \dots}_{=C_3} \quad \text{where}$$

$$A_{22} = \tfrac{1}{21} + \tfrac{1}{45} + \tfrac{1}{77} + \dots = \tfrac{1}{4}(\underbrace{\tfrac{1}{3} - \cancel{\tfrac{1}{7}}}_{=\frac{4}{21}} + \underbrace{\tfrac{1}{5} - \cancel{\tfrac{1}{9}}}_{=\frac{4}{45}} + \underbrace{\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{11}}}_{=\frac{4}{77}} + \dots) = \tfrac{1}{4} \cdot (\tfrac{1}{3} + \tfrac{1}{5}) \ ,$$

$$A_{21} = \tfrac{1}{35} + \tfrac{1}{63} + \tfrac{1}{99} + \dots = \tfrac{1}{2}(\underbrace{\tfrac{1}{5} - \cancel{\tfrac{1}{7}}}_{=\frac{2}{35}} + \underbrace{\cancel{\tfrac{1}{7}} - \cancel{\tfrac{1}{9}}}_{\frac{2}{63}} + \underbrace{\cancel{\tfrac{1}{9}} - \cancel{\tfrac{1}{11}}}_{=\frac{2}{99}} + \dots) = \tfrac{1}{2} \cdot \tfrac{1}{5} \ ,$$

$$-(S_3 - Q_3) = 2\underbrace{(A_3 - \tfrac{1}{7})}_{=A_{33}} - 2\underbrace{(A_2 - \tfrac{1}{5} - \tfrac{1}{21})}_{=A_{32}} + 2\underbrace{(A_1 - \tfrac{1}{3} - \tfrac{1}{15} - \tfrac{1}{35})}_{=A_{33}} -$$

$$- \underbrace{(\tfrac{1}{81} + \tfrac{1}{121} + \dots)}_{=C_4} \quad \text{where}$$

$$A_{33} = \tfrac{1}{27} + \tfrac{1}{55} + \tfrac{1}{91} + \dots = \tfrac{1}{6}(\underbrace{\tfrac{1}{3} - \cancel{\tfrac{1}{9}}}_{=\frac{6}{27}} + \underbrace{\tfrac{1}{5} - \cancel{\tfrac{1}{11}}}_{=\frac{6}{57}} + \underbrace{\tfrac{1}{7} - \cancel{\tfrac{1}{13}}}_{=\frac{6}{91}} + \dots) =$$

$$= \tfrac{1}{6} \cdot (\tfrac{1}{3} + \tfrac{1}{5} + \tfrac{1}{7}) \ ,$$

$$A_{32} = \tfrac{1}{45} + \tfrac{1}{77} + \tfrac{1}{117} + \dots = \tfrac{1}{4}(\underbrace{\tfrac{1}{5} - \cancel{\tfrac{1}{9}}}_{=\frac{4}{45}} + \underbrace{\tfrac{1}{7} - \cancel{\tfrac{1}{11}}}_{=\frac{4}{77}} + \underbrace{\cancel{\tfrac{1}{9}} - \cancel{\tfrac{1}{13}}}_{=\frac{4}{117}} + \dots) = \tfrac{1}{4} \cdot (\tfrac{1}{5} + \tfrac{1}{7}) \ ,$$

$$A_{31} = \tfrac{1}{63} + \tfrac{1}{99} + \tfrac{1}{143} + \dots = \tfrac{1}{2}(\underbrace{\tfrac{1}{7} - \cancel{\tfrac{1}{9}}}_{=\frac{2}{63}} + \underbrace{\cancel{\tfrac{1}{9}} - \cancel{\tfrac{1}{11}}}_{=\frac{2}{99}} + \underbrace{\cancel{\tfrac{1}{11}} - \cancel{\tfrac{1}{13}}}_{=\frac{2}{143}} + \dots) = \tfrac{1}{2} \cdot \tfrac{1}{7} \ ,$$

and so on. In general

$$(-1)^n \cdot (S_n - Q_n) = 2 \cdot A_{nn} - 2 \cdot A_{n\,n-1} + \dots + (-1)^{n-1} \cdot 2 \cdot A_{n1} + (-1)^n \cdot C_{n+1}$$

$$\text{where } A_{nm} = \frac{1}{2m}(\frac{1}{2n+1} + \frac{1}{2n-1} + \dots + \frac{1}{2(n-m)+3}) \ ,$$

$$C_n = \frac{1}{(2n+1)^2} + \frac{1}{(2n+3)^2} + \dots \quad \text{for} \quad 1 \le m \le n \ .$$

Simple calculation give us the following inequalities and limits

$$A_{n1} \le A_{n2} \le \cdots \le A_{n\,n-1} \le A_{nn} \quad \text{and}$$
$$0 \le A_{nn} - A_{n\,n-1} + \cdots + (-1)^{n-1} \cdot A_{n1} \le A_{nn} \le A_n \to 0$$

$$C_n \to 0 \ .$$

From

$$|S_n - Q_n| \le 2 \cdot \underbrace{|A_{nn} - A_{n\,n-1} + \cdots + (-1)^{n-1} \cdot A_{n1}|}_{\le |A_{nn}| \le A_n \to 0} + \underbrace{|C_{n+1}|}_{\to 0}$$

we have

$$S_n - Q_n \to 0$$

and hence the limit of both partial sums must approach the same value and

$$A_0 - 2A = s^2 \ . \tag{3}$$

2.3. *Sums T_n and Q_n*

Inequalities between sums T_n and Q_n

$$T_1 = B_1 - B_2 \le Q_1 \le B_1 - B_2 + B_3 = T_2 \ ,$$
$$T_3 = B_1 - B_2 + B_3 - B_4 \le Q_3 \le$$
$$\le B_1 - B_2 + B_3 - B_4 + B_5 - B_6 + B_7 = T_6 \ ,$$
$$T_5 = B_1 - B_2 + B_3 - B_4 + B_5 - B_6 \le Q_5 \le$$
$$\le B_1 - B_2 + B_3 - B_4 + B_5 - B_6 + B_7 - B_8 + B_9 - B_{10} + B_{11} = T_{10} \ ,$$

generally

$$T_n = B_1 - B_2 + \cdots + (-1)^n \cdot B_{n+1} \le Q_n \le B_1 - B_2 + \cdots + B_{2n+1} = T_{2n}$$

for any odd n give us

$$\lim T_n \le \lim Q_n \le \lim T_n, \quad 2A \le s^2 \le 2A \quad \text{and} \quad 2A = s^2 \ . \tag{4}$$

2.4. *Conclusion*

As we remind (3) and (4) we have

$$s^2 = A_0 - 2A = A_0 - s^2 \ , \quad \text{hence} \quad A_0 = 2s^2 = \frac{\pi^2}{8}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = A_0 = \frac{\pi^2}{8} \ .$$

The investigated series can be divided into odd and even terms

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{\pi^2}{8} + \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and we can conclude

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \ .$$

References

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