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MEASURE OF PLANES INTERSECTING A CONVEX BODY

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In the present paper a representation for the measure of planes intersecting a convex body, using stochastic approximation of the body was found. The representation was found in terms of the normal curvatures of the surface of the body and a flag density of the measure.

Keywords: integral geometry; stochastic approximation; flag densities.

1. Introduction

Let $\mathbf{E}$ be the space of planes in $\mathbf{R}^3$. We consider locally finite signed measures μ in the space $\mathbf{E}$, which possess densities with respect to the standard Euclidean motion invariant measure,
i. e. (see. [**?**])

$$\mu(de) = h(e)\, de. \tag{1}$$

Recall that an element de of the standard measure is written as $de = dp \cdot d\xi$, where (p, ξ) is the usual parametrization of a plane e: p is the distance of e from the origin O; $\xi \in \mathrm{S}^2$ is the direction normal to e, $d\xi$ is an element of solid angle of the unit sphere S^2. Where appropriate we write $h(e) = h(p, \xi)$.

The concept of a flag in $\mathbf{R}^3$ which naturally emerges in Combinatorial integral geometry will be of basic importance below. A detailed account of this concept is in [**?**]. We repeat the definition.

A flag is a triad $f = (P, g, e)$, where P is a point in $\mathbf{R}^3$ called the *l*ocation of f, g is a line containing the point P, and e is a plane containing g. There are two equivalent representations of a flag:

$$f = f(P, \Omega, \Phi) \;\text{ or }\; f = f(P, \omega, \varphi),$$

where Ω is the spatial direction of g in $\mathbf{R}^3$, Φ is the rotation of e around g, ω is the normal of e, and φ is the planar direction of g in e. The range of Ω and ω is

$\mathcal{E}_2$, the standard elliptic 2–space which can be obtained from the unit sphere by identification of the antipodal points ([?]), ϕ and φ belong to $\mathcal{E}_1$.

We introduce the following function in the space of flags (a flag function)

$$\rho(f) = \rho(P, \omega, \varphi) = \int_{\mathcal{E}_2} \cos^2(\varphi - \psi)\, h_{[P]}(\xi)\, d\xi. \tag{2}$$

Here $[P]$ is the bundle of planes containing the point $P \in \mathbf{R}^3$, $h_{[P]}(\xi)$ is the restriction of h onto $[P]$, ψ is the direction of the projection of ξ into the plane of the flag f. The notation $h_{[P]}(\xi)$ is reasonable since ξ completely determines a plane from $[P]$. Clearly, the integral (2) does not depend on the choice of the reference point on the plane of the flag f. The function ρ defined on the space of flags $\mathcal{F}$ we call flag density. The concept of a flag density was introduced and systematically employed by R. V. Ambartzumian (see [?],[?]).

Note, that in [?] (see also [?] and [?]) (2) was considered as an integral equation and by integral geometry methods was recovered h from a given ρ.

Let $\mathbf{B}$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature at every point of $\partial\mathbf{B}$. By $[\mathbf{B}]$ we denote the set of planes intersecting $\mathbf{B}$. Let $s(\omega)$ be a point on $\partial\mathbf{B}$ whose outer normal is ω. By $k_1(\omega), k_2(\omega)$ we denote the principal normal curvatures of $\partial\mathbf{B}$ at $s(\omega)$ and let $k(\omega, \varphi)$ be the normal curvature in the direction φ at the point $s(\omega)$ of $\partial\mathbf{B}$, φ is measured from the first principal direction.

The main result of the paper is the following.

Theorem 1.1. *Let μ be a signed measure on $\mathbf{E}$, possessing a density $h(e)$. For any sufficiently smooth convex body $\mathbf{B}$ we have the following representation:*

$$\mu([\mathbf{B}]) = (2\pi^2)^{-1} \int_{\mathrm{S}^2} \int_0^{2\pi} \rho(s(\omega), \omega, \varphi) \frac{\sqrt{k_1(\omega) k_2(\omega)}}{k^2(\omega, \varphi)} d\varphi\, d\omega, \tag{3}$$

where ρ is the flag density of μ defined by (2).

If $h(e) \equiv 1$ (the case of Euclidean motion invariant measure μ_{inv}) from (3) we obtain the Minkowski formula (see [?])

$$\mu_{\mathrm{inv}}([\mathbf{B}]) = \frac{1}{2} \int_{\mathrm{S}^2} \left(\frac{1}{k_1(\omega)} + \frac{1}{k_2(\omega)} \right) d\omega. \tag{4}$$

2. Preliminary Representations

In [?] R. V. Ambartzumian has indicated the existence of the so-called *flag-representation* for width functions of convex bodies in $\mathbf{R}^3$ using some "standard" flag-representation for width functions of polyhedra.

Let $H(\xi)$ be the width function in direction ξ of a convex body $\mathbf{B}$. Then (see [?])

$$H(\xi) = \int_{\mathrm{S}^1 \times \mathrm{S}^2} \sin^2 \alpha(\xi, \Omega, \Phi)\, m(d\Omega, d\Phi), \tag{5}$$

where S^i is the unit sphere in $\mathbf{R}^{i+1}$, $i = 1, 2$, Ω, $\xi \in \mathrm{S}^2$, m is a measure in $\mathrm{S}^1 \times \mathrm{S}^2$, α is the angle between $\Omega \in \mathrm{S}^2$ and the trace $e_\xi \cap e(\Omega, \Phi)$, e_ξ is a plane normal to ξ, and $e(\Omega, \Phi)$ is the plane of the so-called free flag $f = f(\Omega, \Phi)$ (one can consider that the location of a free flag is the origin of $\mathbf{R}^3$).
The representation (5) fails to be unique (there are many m for given H).
Note, that if μ is a translation invariant measure in $\mathbf{E}$ with the form $d\mu = dp \times \delta_\xi$, where δ_ξ is a delta measure concentrated on the direction ξ, then we have $H(\xi) = \mu([\mathbf{B}])$.

Let $\mathbf{K} \subset \mathbf{R}^3$ be a convex polyhedron and $e \in [\mathbf{K}] \subset \mathbf{E}$. We consider the intersection $e \cap \mathbf{K}$ which is a bounded convex polygon whose vertices correspond to the edges of $\mathbf{K}$ actually hit by e. The fact, that the sum of outer angles of $e \cap \mathbf{K}$ equals 2π we write in the form

$$\sum_i \alpha_i(e)\, I_{[L_i]}(e) = 2\pi I_{[\mathbf{K}]}. \tag{6}$$

Here L_i is an edge of $\mathbf{K}$, $\alpha_i(e)$ is the outer angle of $e \cap \mathbf{K}$ correspond to vertex $e \cap L_i$, and summation is by all edges of $\mathbf{K}$.
In [?], by integration of (6) with respect to $d\mu = dp \times m(d\xi)$ (a translation invariant measure) some "standard" flag-representation for the width function of a polyhedron $\mathbf{K}$ was found. In [?], using approximation by polyhedrons, a new representation for the width functions of convex bodies was obtained. In [?], using stochastic approximation (Voronoi's approximation) of smooth convex bodies by polyhedrons, for translation invariant measures representation (3) was obtained.
Now we integrate (6) with respect to $\mu(de) = h(e)de$, where h is a continuous function defined on $\mathbf{E}$. We have

$$2\pi\mu([\mathbf{K}]) = \sum_i \int_{[L_i]} \alpha_i(e)\, h(e)\, de = \sum_i \int_{[L_i]} \alpha_i(e)\, h(p,\xi)\, dp\, d\xi$$
$$= \sum_i \int_{[L_i]} \alpha_i(e) h(x,\xi) \mid \cos(\widehat{\xi,\Omega_i}) \mid dx d\xi = \sum_i \int_{L_i} \int_{\mathcal{E}_2} \alpha_i(e) h(x,\xi) \mid \cos(\widehat{\xi,\Omega_i}) \mid d\xi dx. \tag{7}$$

Here $\widehat{\xi,\Omega_i}$ is the angle between ξ and Ω_i, where Ω_i is the direction of the edge L_i. Also, here we use the following well known fact from integral geometry

$$de = dp\, d\xi = \mid \cos(\widehat{\xi,\Omega_i}) \mid \; dx\, d\xi, \tag{8}$$

where x is the intersection point $e \cap L_i$ and dx is one dimensional Lebesque in L_i. Using standard formulae of spherical trigonometry we get (see [?])

$$\alpha_i(e) \; \mid \cos(\widehat{\xi,\Omega_i}) \mid = \int_{A_i} \sin^2 \alpha(\xi, \Omega_i, \Phi)\, d\Phi, \tag{9}$$

where A_i is the exterior dihedral angle of the edge L_i (see also (5)). After substitution (9) into (7) we obtain

$$2\pi\mu([\mathbf{K}]) = \sum_i \int_{L_i} \int_{A_i} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_i, \Phi)\, h(x,\xi) d\xi\, d\Phi\, dx. \tag{10}$$

3. Stochastic Approximation

Let $\mathbf{B}$ be a sufficiently smooth (three times continuously differentiable) convex body in $\mathbf{R}^3$. We assume that the Gaussian curvature of $\partial\mathbf{B}$ is everywhere positive. Hence the Gauss map of $\partial\mathbf{B}$ onto the unit sphere S^2 is a homeomorphism.

We throw n independent points $P_1, \dots, P_n$ onto S^2 with the same distribution P. Let $d\mathrm{P} = f(\omega)d\omega$, where $f(\omega) > 0$ is continuous, $d\omega$ is an area element on S^2. On $\partial\mathbf{B}$ by $P_1^*, \dots, P_n^*$ we denote the images of the points $P_1, \dots, P_n$ by the inverse to the Gauss map. Denote by $\mathbf{K_n}(P_1^*, \dots, P_n^*)$ the convex hull of the points $P_1^*, \dots, P_n^*$. According to (10), $\mu([\mathbf{K_n}(P_1^*, \dots, P_n^*)])$ can be represented in the form

$$2\pi\mu([\mathbf{K_n}]) = \sum_{i<j \;\; i,j=1}^{n} I_D(i,j) \int_{L_{ij}} \int_{A_{ij}} \int_{\mathcal{E}^2} \sin^2 \alpha(\xi, \Omega_{ij}, \Phi)\, h(x,\xi) d\xi\, d\Phi\, dx. \quad (11)$$

Here Ω_{ij} is the direction of $\overrightarrow{P_i^* P_j^*}$, L_{ij} is the edge $P_i^* P_j^*$, A_{ij} is the exterior dihedral angle of the edge $P_i^* P_j^*$, D is the set of all pairs (i,j) corresponding to the edge. We average both sides of (11) with respect to the sequences $(P_1^*, \dots, P_n^*)$. Since $f(\omega) > 0$, in the limit $(n \to \infty)$ in the left-hand side we obtain $\mu([\mathbf{B}])$. By symmetry we have

$$2\pi\mu([\mathbf{B}]) = \lim_{n\to\infty} \binom{n}{2} \int_{(\mathrm{S}^2)^2} \left[\int_{(\mathrm{S}^2)^{n-2}} I_D(1,2) \right.$$
$$\left. \times \left[\int_{L_{12}} \int_{A_{12}} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{12}, \Phi)\, h(x,\xi) d\xi\, d\Phi\, dx \right] dP_3 \cdots dP_n \right] dP_1\, dP_2. \quad (12)$$

Taking P_1 as the pole, P_2 can be described by spherical coordinates (ν, φ) with respect to P_1. Also, P_2 can be described by coordinates (l, φ), where $l = |P_1 P_2|$. We have

$$dP_2 = f(\omega)\, d\omega = f(\nu, \varphi)\, \sin\nu\, d\nu\, d\varphi = f(l, \varphi)\, l\, dl\, d\varphi. \quad (13)$$

Let $e(\Omega_{l\varphi}, \Phi)$ be the plane passing through P_1^*, P_2^* and rotated around $\Omega_{l\varphi} = \overrightarrow{P_1^* P_2^*}$ by angle Φ. For $e(\Omega_{l\varphi}, 0)$ we take the plane that is perpendicular to the plane passing through ω and $\Omega_{l\varphi}$. By L^* we denote the segment $P_1^* P_2^*$ and let $l^* = |P_1^* P_2^*|$.

In this paper we consider the case of the uniform distribution, i.e. $f(\omega) = (S_o C(\omega))^{-1}$, where S_o is the total area of the surface of $\mathbf{B}$, $C(\omega)$ is the Gaussian curvature at the point on $\partial\mathbf{B}$ with normal ω. The plane $e(\Omega_{l\varphi}, \Phi)$ divides $\partial\mathbf{B}$ into two parts and by $S(\Phi, l)$ we denote the area of the smaller part $\partial\mathbf{B}_1(\Phi, l)$. Applying Fubini's theorem in the inner integral of (12), we obtain

$$2\pi\mu([\mathbf{B}]) = \lim_{n\to\infty} \binom{n}{2} \int_{(\mathrm{S}^2)^2} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(1 - \frac{S(\Phi,l)}{S_o}\right)^{n-2} + \left(\frac{S(\Phi,l)}{S_o}\right)^{n-2} \right] \right.$$
$$\left. \times \left[\int_{L^*} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{l\varphi}, \Phi)\, h(x,\xi) d\xi\, dx \right] d\Phi \right] \frac{l\, dl\, d\varphi\, d\omega}{S_o^2 C(\omega) C(l,\varphi)}. \quad (14)$$

The sum in the square brackets of (14) is the probability that segment $P_1^* P_2^*$ is an edge and $e(\Omega_{l\varphi}, \Phi)$ belongs to the exterior dihedral angle of the edge.

Since $\frac{S(\Phi,l)}{S_o} \leq \frac{1}{2}$, we have

$$\lim_{n\to\infty} \left| \binom{n}{2} \int_{(\mathrm{S}^2)^2} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{S(\Phi,l)}{S_o} \right)^{n-2} \left[\int_{L^*} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{l\varphi}, \Phi)\, h(x,\xi) d\xi\, dx \right] d\Phi \right] \times \frac{l\, dl\, d\varphi\, d\omega}{S_o^2 C(\omega) C(l,\varphi)} \right| \leq \lim_{n\to\infty} A \binom{n}{2} \left(\frac{1}{2}\right)^{n-2} = 0, \tag{15}$$

where A is a constant. In a similar manner one can prove that the domain of variation of Φ and l can be taken arbitrarily small. Thus

$$2\pi\mu([\mathbf{B}]) = \lim_{n\to\infty} \binom{n}{2} \int_{(\mathrm{S}^2)} \int_0^{2\pi} \int_0^{l_0} \int_{-\Phi_0}^{\Phi_0} \left(1 - \frac{S(\Phi,l)}{S_o}\right)^{n-2} \times \left[\int_{L^*} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{l\varphi}, \Phi)\, h(x,\xi) d\xi\, dx \right] l\, d\Phi\, dl \frac{d\varphi\, d\omega}{S_o^2 C(\omega) C(l,\varphi)}, \tag{16}$$

where l_0 and Φ_0 are arbitrarily small fixed numbers. From the regularity of the surface $\partial\mathbf{B}$ we obtain the Taylor expansion

$$S(\Phi,l) = l\, S'_l(0,0) + \Phi S'_\Phi(0,0) + \frac{l^2}{2} S''_{ll}(0,0) + l\Phi S''_{l\Phi}(0,0) + \frac{\Phi^2}{2} S''_{\Phi\Phi}(0,0) + R(\Phi,l), \tag{17}$$

where $R(\Phi,l) = o(l^2 + \Phi^2)$. Here all functions continuously depend on l and Φ, as well as on ω and φ. Below, we will see, that $S'_l(0,0) = S'_\Phi(0,0) = 0$.
Using the mean value theorem we find

$$\int_{L^*} \int_{\mathcal{E}^2} \sin^2 \alpha(\xi, \Omega_{l\varphi}, \Phi)\, h(x,\xi) d\xi\, dx = l^* \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{l\varphi}, \Phi)\, h(x_0,\xi) d\xi, \tag{18}$$

where x_0 is a point from the segment L^* and $l^* = |L^*|$.
After substitution (18) in (16) and a change of variables $u = l\sqrt{n}$, $v = \Phi\sqrt{n}$ we get

$$2\pi\mu([\mathbf{B}]) = \lim_{n\to\infty} \binom{n}{2} \int_{(\mathrm{S}^2)} \int_0^{2\pi} \left[\int_0^{l_0\sqrt{n}} \int_{-\Phi_0\sqrt{n}}^{\Phi_0\sqrt{n}} \left(1 - \frac{S(\frac{v}{\sqrt{n}}, \frac{u}{\sqrt{n}})}{S_o}\right)^{n-2} \times \left[\int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{\frac{u}{\sqrt{n}}\varphi}, \frac{v}{\sqrt{n}})\, h(x_0,\xi) d\xi \right] \frac{u(u \cdot b(\omega,\varphi)) du\, dv}{n^2 C(\frac{v}{\sqrt{n}}, \varphi)} \right] \frac{d\varphi\, d\omega}{S_o^2 C(\omega)}, \tag{19}$$

where $l^* = l \cdot b(\omega,\varphi) + o(l)$.
One can interchange the limit and the integration operations. Substitution (17) in (19) and making use of

$$\int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{\frac{u}{\sqrt{n}}\varphi}, \frac{v}{\sqrt{n}})\, h(x_0,\xi) d\xi \to \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \omega, \varphi_1)\, h_{[P^*(\omega)]}(\xi) d\xi \tag{20}$$

almost everywhere when $n \to \infty$, where $\alpha(\xi,\omega,\varphi_1)$ is the angle between the direction φ_1 in the plane e_ω and the intersection of e_ξ with the plane e_ω, $[P^*(\omega)]$ is the bundle of planes containing the point $P^*(\omega) \in \partial\mathbf{B}$ with normal ω, we obtain

$$2\pi\mu([\mathbf{B}]) = \int_{(\mathrm{S}^2)} \int_0^{2\pi} \left[\int_0^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2} S''_{ll}(0,0) - uv S''_{l\Phi}(0,0) - \frac{v^2}{2} S''_{\Phi\Phi}(0,0) \right) \right.$$
$$\left. \times\, u^2 du\, dv \right] \left[\int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \omega, \varphi_1)\, h_{[P^*(\omega)]}(\xi) d\xi \right] \frac{b(\omega,\varphi)}{2C(\omega)} d\varphi\, d\omega, \tag{21}$$

where $b(\omega,\varphi) = \lim_{l\to 0} \frac{l^*}{l}$.

4. A Representation Obtained by Stochastic Approximation

It follows from (21) that the final representation for $\mu([\mathbf{B}])$ depends on values of $S''_{ll}(0,0)$, $S''_{l\Phi}(0,0)$, $S''_{\Phi\Phi}(0,0)$ which are functions of ω and φ (see (21)). It was proved in [**?**], that:

$S'_l(0,0)$, $S'_\Phi(0,0)$, $S''_{ll}(0,0)$, $S''_{l\Phi}(0,0)$, $S''_{\Phi\Phi}(0,0)$ *depend only on derivatives of at most order of two of the surface* $\partial\mathbf{B}$ *at the point* P^* *whose outer normal is* ω.
Hence the corresponding calculation we can do for the osculating paraboloid of $\partial\mathbf{B}$ at the point $P^*(\omega)$ whose outer normal is ω. In [**?**] the following expressions for the derivatives in terms of the normal curvatures of $\partial\mathbf{B}$ at the point $P^*(\omega)$ were found:

$$S'_l(0,0) = 0, \quad S'_\Phi(0,0) = 0, \quad S''_{ll}(0,0) = \frac{\pi\sqrt{k_1k_2}\,r^2(\varphi)(k_2^3\cos^2\varphi + k_1^3\sin^2\varphi)}{2A^4},$$
$$S''_{l\Phi}(0,0) = \frac{\pi\sqrt{k_1k_2}\sin 2\varphi(k_2-k_1)}{2A^3}, \quad S''_{\Phi\Phi}(0,0) = \frac{2\pi\sqrt{k_1k_2}\,r(\varphi)}{A^2}, \tag{22}$$

where k_i, $i = 1,2$ are the main normal curvatures, $r(\varphi) = k_1^{-1}\cos^2\varphi + k_2^{-1}\sin^2\varphi$ is the radius of the normal curvature in the direction φ at the point $P^*(\omega)$ of $\partial\mathbf{B}$ and $A = \sqrt{k_2^2\cos^2\varphi + k_1^2\sin^2\varphi}$. Also, in [**?**] was found that (see (21))

$$b(\omega,\varphi) = \sqrt{\frac{\cos^2\varphi}{k_1^2} + \frac{\sin^2\varphi}{k_2^2}} \quad \text{and} \quad \tan\varphi_1 = \tan\varphi\,\frac{k_1}{k_2}. \tag{23}$$

Substituting (22) and (23) into (21) we get

$$\mu([\mathbf{B}]) = (2\pi^2)^{-1} \int_{(\mathrm{S}^2)} \int_0^{2\pi} \left[\int_{\mathcal{E}^2} \sin^2\alpha(\xi,\omega,\varphi)\, h_{[P^*(\omega)]}(\xi) d\xi \right] \frac{\sqrt{k_1k_2}}{k^2(\omega,\varphi)} d\varphi\, d\omega, \tag{24}$$

where $k(\omega,\varphi)$ is the normal curvature in the direction φ at the point $P^*(\omega)$ of $\partial\mathbf{B}$. Taking into account that

$$\sin^2\alpha(\xi,\omega,\varphi) = \cos^2(\varphi - \psi), \tag{25}$$

where ψ is the direction of the projection of ξ into the plane with normal ω, we get (3).
Theorem 1 is proved.

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