MEASURE OF PLANES INTERSECTING
A CONVEX BODY

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In the present paper a representation for the measure of planes intersecting a convex body, using stochastic approximation of the body was found. The representation was found in terms of the normal curvatures of the surface of the body and a flag density of the measure.

Keywords: integral geometry; stochastic approximation; flag densities.

1. Introduction
Let \( E \) be the space of planes in \( \mathbb{R}^3 \). We consider locally finite signed measures \( \mu \) in the space \( E \), which possess densities with respect to the standard Euclidean motion invariant measure, i.e. (see. [2])

\[
\mu(de) = h(e) \, de.
\] (1)

Recall that an element \( de \) of the standard measure is written as \( de = dp \cdot d\xi \), where \((p, \xi)\) is the usual parametrization of a plane \( e \): \( p \) is the distance of \( e \) from the origin \( O \); \( \xi \in S^2 \) is the direction normal to \( e \). \( d\xi \) is an element of solid angle of the unit sphere \( S^2 \). Where appropriate we write \( h(e) = h(p, \xi) \).

The concept of a flag in \( \mathbb{R}^3 \) which naturally emerges in Combinatorial integral geometry will be of basic importance below. A detailed account of this concept is in [?]. We repeat the definition.

A flag is a triad \( f = (P, g, e) \), where \( P \) is a point in \( \mathbb{R}^3 \) called the location of \( f \), \( g \) is a line containing the point \( P \), and \( e \) is a plane containing \( g \). There are two equivalent representations of a flag:

\[
f = f(P, \Omega, \Phi) \text{ or } f = f(P, \omega, \varphi),
\]

where \( \Omega \) is the spatial direction of \( g \) in \( \mathbb{R}^3 \), \( \Phi \) is the rotation of \( e \) around \( g \), \( \omega \) is the normal of \( e \), and \( \varphi \) is the planar direction of \( g \) in \( e \). The range of \( \Omega \) and \( \omega \) is
$\mathcal{E}_2$, the standard elliptic 2–space which can be obtained from the unit sphere by identification of the antipodal points \([\phi])$, $\phi$ and $\varphi$ belong to $\mathcal{E}_1$.

We introduce the following function in the space of flags (a flag function)

$$\rho(f) = \rho(P, \omega, \varphi) = \int_{\mathcal{E}_2} \cos^2(\varphi - \psi) h_{[P]}(\xi) \, d\xi.$$  \hspace{1cm} (2)

Here $[P]$ is the bundle of planes containing the point $P \in \mathbb{R}^3$, $h_{[P]}(\xi)$ is the restriction of $h$ onto $[P]$, $\psi$ is the direction of the projection of $\xi$ into the plane of the flag $f$. The notation $h_{[P]}(\xi)$ is reasonable since $\xi$ completely determines a plane from $[P]$. Clearly, the integral (2) does not depend on the choice of the reference point on the plane of the flag $f$. The function $\rho$ defined on the space of flags $\mathcal{F}$ we call flag density. The concept of a flag density was introduced and systematically employed by R. V. Ambartzumian (see [?,?], [?]).

Note, that in [?] (see also [?] and [?]) (2) was considered as an integral equation and by integral geometry methods was recovered $h$ from a given $\rho$.

Let $B$ be a convex body with sufficiently smooth boundary and with positive Gaussian curvature at every point of $\partial B$. By $[B]$ we denote the set of planes intersecting $B$. Let $s(\omega)$ be a point on $\partial B$ whose outer normal is $\omega$. By $k_1(\omega), k_2(\omega)$ we denote the principal normal curvatures of $\partial B$ at $s(\omega)$ and let $k(\omega, \varphi)$ be the normal curvature in the direction $\varphi$ at the point $s(\omega)$ of $\partial B$, $\varphi$ is measured from the first principal direction.

The main result of the paper is the following.

**Theorem 1.1.** Let $\mu$ be a signed measure on $\mathcal{E}$, possessing a density $h(e)$. For any sufficiently smooth convex body $B$ we have the following representation:

$$\mu([B]) = \frac{1}{2} \int_{S^2} \rho(s(\omega), \omega, \varphi) \frac{\sqrt{k_1(\omega)k_2(\omega)}}{k^2(\omega, \varphi)} \, d\varphi \, d\omega,$$  \hspace{1cm} (3)

where $\rho$ is the flag density of $\mu$ defined by (2).

If $h(e) \equiv 1$ (the case of Euclidean motion invariant measure $\mu_{\text{inv}}$) from (3) we obtain the Minkowski formula (see [?])

$$\mu_{\text{inv}}([B]) = \frac{1}{2} \int_{S^2} \left( \frac{1}{k_1(\omega)} + \frac{1}{k_2(\omega)} \right) \, d\omega.$$  \hspace{1cm} (4)

**2. Preliminary Representations**

In [?] R. V. Ambartzumian has indicated the existence of the so-called flag-representation for width functions of convex bodies in $\mathbb{R}^3$ using some ”standard” flag-representation for width functions of polyhedra.

Let $H(\xi)$ be the width function in direction $\xi$ of a convex body $B$. Then (see [?])

$$H(\xi) = \int_{S^1 \times S^2} \sin^2 \alpha(\xi, \Omega, \Phi) \, m(d\Omega, d\Phi),$$  \hspace{1cm} (5)
where $S^i$ is the unit sphere in $\mathbb{R}^{i+1}$, $i = 1, 2$, $\Omega, \xi \in S^2$, $m$ is a measure in $S^1 \times S^2$, $\alpha$ is the angle between $\Omega \in S^2$ and the trace $e_\xi \cap e(\Omega, \Phi)$, $e_\xi$ is a plane normal to $\xi$, and $e(\Omega, \Phi)$ is the plane of the so-called free flag $f = f(\Omega, \Phi)$ (one can consider that the location of a free flag is the origin of $\mathbb{R}^3$).

The representation (5) fails to be unique (there are many $m$ for given $H$).

Note, that if $\mu$ is a translation invariant measure in $E$ with the form $d\mu = dp \times \delta_\xi$, where $\delta_\xi$ is a delta measure concentrated on the direction $\xi$, then we have $H(\xi) = \mu([B])$.

Let $K \subset \mathbb{R}^3$ be a convex polyhedron and $e \in [K] \subset E$. We consider the intersection $e \cap K$ which is a bounded convex polygon whose vertices correspond to the edges of $K$ actually hit by $e$. The fact, that the sum of outer angles of $e \cap K$ equals $2\pi$ we write in the form

$$\sum_i \alpha_i(e) I_{[L_i]}(e) = 2\pi I_K.$$  

(6)

Here $L_i$ is an edge of $K$, $\alpha_i(e)$ is the outer angle of $e \cap K$ correspond to vertex $e \cap L_i$, and summation is by all edges of $K$.

In [7], by integration of (6) with respect to $d\mu = dp \times m(d\xi)$ (a translation invariant measure) some "standard" flag-representation for the width function of a polyhedron $K$ was found. In [8], using approximation by polyhedrons, a new representation for the width functions of convex bodies was obtained. In [7], using stochastic approximation (Voronoi’s approximation) of smooth convex bodies by polyhedrons, for translation invariant measures representation (3) was obtained.

Now we integrate (6) with respect to $\mu(de) = h(e)de$, where $h$ is a continuous function defined on $E$. We have

$$2\pi \mu([K]) = \sum_i \int_{[L_i]} \alpha_i(e) h(e) de = \sum_i \int_{[L_i]} \alpha_i(e) h(p, \xi) dp \, d\xi$$

$$= \sum_i \int_{[L_i]} \alpha_i(e) h(x, \xi) | \cos(\xi, \Omega_i) | \, dx \, d\xi = \sum_i \int_{L_i} \int_{\mathcal{E}_2} \alpha_i(e) h(x, \xi) | \cos(\xi, \Omega_i) | \, d\xi \, dx.$$  

(7)

Here $\xi, \Omega_i$ is the angle between $\xi$ and $\Omega_i$, where $\Omega_i$ is the direction of the edge $L_i$.

Also, here we use the following well known fact from integral geometry

$$de = dp \, d\xi = | \cos(\xi, \Omega_i) | \, dx \, d\xi,$$  

(8)

where $x$ is the intersection point $e \cap L_i$ and $dx$ is one dimensional Lebesgue in $L_i$.

Using standard formulae of spherical trigonometry we get (see [7])

$$\alpha_i(e) | \cos(\xi, \Omega_i) | = \int_{A_i} \sin^2 \alpha(\xi, \Omega_i, \Phi) \, d\Phi,$$  

(9)

where $A_i$ is the exterior dihedral angle of the edge $L_i$ (see also (5)). After substitution (9) into (7) we obtain

$$2\pi \mu([K]) = \sum_i \int_{L_i} \int_{A_i} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_i, \Phi) h(x, \xi) d\xi \, d\Phi \, dx.$$  

(10)
3. Stochastic Approximation

Let \( B \) be a sufficiently smooth (three times continuously differentiable) convex body in \( \mathbb{R}^3 \). We assume that the Gaussian curvature of \( \partial B \) is everywhere positive. Hence the Gauss map of \( \partial B \) onto the unit sphere \( S^2 \) is a homeomorphism.

We throw \( n \) independent points \( P_1, \ldots, P_n \) onto \( S^2 \) with the same distribution \( P \). Let \( dP = f(\omega)d\omega \), where \( f(\omega) > 0 \) is continuous, \( d\omega \) is an area element on \( S^2 \). On \( \partial B \) by \( P_1^*, \ldots, P_n^* \) we denote the images of the points \( P_1, \ldots, P_n \) by the inverse to the Gauss map. Denote by \( K_n(P_1^*, \ldots, P_n^*) \) the convex hull of the points \( P_1^*, \ldots, P_n^* \).

According to (10), \( \mu([K_n(P_1^*, \ldots, P_n^*)]) \) can be represented in the form

\[
2\pi \mu([K_n]) = \sum_{i<j}^{n} \int_{L_{ij}} \int_{A_{ij}} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{ij}, \Phi) h(x, \xi) d\xi d\Phi dx. \quad (11)
\]

Here \( \Omega_{ij} \) is the direction of \( P_1^*P_2^* \), \( L_{ij} \) is the edge \( P_i^*P_j^* \), \( A_{ij} \) is the exterior dihedral angle of the edge \( P_i^*P_j^* \), \( D \) is the set of all pairs \((i, j)\) corresponding to the edge. We average both sides of (11) with respect to the sequences \( (P_1^*, \ldots, P_n^*) \). Since \( f(\omega) > 0 \), in the limit \((n \to \infty)\) in the left-hand side we obtain \( \mu([B]) \). By symmetry we have

\[
2\pi \mu([B]) = \lim_{n \to \infty} \left( \frac{n}{2} \right) \int_{(S^2)^2} \left[ \int_{(S^2)^{n-2}} I_D(1, 2) \right. \\

\times \left. \int_{L_{12}} \int_{A_{12}} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{12}, \Phi) h(x, \xi) d\xi d\Phi dx \right] dP_3 \cdots dP_n dP_1 dP_2. \quad (12)
\]

Taking \( P_1 \) as the pole, \( P_2 \) can be described by spherical coordinates \((\nu, \varphi)\) with respect to \( P_1 \). Also, \( P_2 \) can be described by coordinates \((l, \varphi)\), where \( l = |P_1P_2| \). We have

\[
dP_2 = f(\omega) d\omega = f(\nu, \varphi) \sin \nu d\nu d\varphi = f(l, \varphi) l dl d\varphi. \quad (13)
\]

Let \( e(\Omega_{l\varphi}, \Phi) \) be the plane passing through \( P_1^*, P_2^* \) and rotated around \( \Omega_{l\varphi} = P_1^*P_2^* \) by angle \( \Phi \). For \( e(\Omega_{l\varphi}, 0) \) we take the plane that is perpendicular to the plane passing through \( \omega \) and \( \Omega_{l\varphi} \). By \( L^* \) we denote the segment \( P_1^*P_2^* \) and let \( l^* = |P_1^*P_2^*| \).

In this paper we consider the case of the uniform distribution, i.e. \( f(\omega) = (S_o C(\omega))^{-1}, \) where \( S_o \) is the total area of the surface of \( B \), \( C(\omega) \) is the Gaussian curvature at the point on \( \partial B \) with normal \( \omega \). The plane \( e(\Omega_{l\varphi}, \Phi) \) divides \( \partial B \) into two parts and by \( S(\Phi, l) \) we denote the area of the smaller part \( \partial B_1(\Phi, l) \).

Applying Fubini’s theorem in the inner integral of (12), we obtain

\[
2\pi \mu([B]) = \lim_{n \to \infty} \left( \frac{n}{2} \right) \int_{(S^2)^2} \left[ \int_{L^*} \int_{\mathcal{E}_2} \alpha(\xi, \Omega_{l\varphi}, \Phi) h(x, \xi) d\xi dx \right] d\Phi \\

\times \left[ \int_{L^*} \int_{\mathcal{E}_2} \sin^2 \alpha(\xi, \Omega_{l\varphi}, \Phi) h(x, \xi) d\xi dx \right] d\Phi \frac{l dl d\varphi d\omega}{S^2_o C(\omega) C(l, \varphi)}. \quad (14)
\]

The sum in the square brackets of (14) is the probability that segment \( P_1^*P_2^* \) is an edge and \( e(\Omega_{l\varphi}, \Phi) \) belongs to the exterior dihedral angle of the edge.
Since \( \frac{S(\Phi,l)}{S_o} \leq \frac{1}{2} \), we have

\[
\lim_{n \to \infty} \left( \frac{n}{2} \right) \int_{(S^2)^2} \left[ \int_{-\hat{\Phi}}^{\hat{\Phi}} \left( \frac{S(\Phi,l)}{S_o} \right)^{n-2} \left[ \int_{L^*} \int_{E_2} \sin^2(\xi, \Omega_{l\varphi}, \Phi) h(x, \xi) d\xi \ dx \right] d\Phi \right] \times \frac{l \ d\varphi \ d\omega}{S_o^2 C(\omega) C(l, \varphi)} \leq \lim_{n \to \infty} A \left( \frac{n}{2} \right) \left( \frac{1}{2} \right)^{n-2} = 0, \tag{15}\]

where \( A \) is a constant. In a similar manner one can prove that the domain of variation of \( \Phi \) and \( l \) can be taken arbitrarily small. Thus

\[
2\pi \mu([B]) = \lim_{n \to \infty} \left( \frac{n}{2} \right) \int_{(S^2)^2} \int_{0}^{2\pi} \int_{0}^{\phi_0} \left( 1 - \frac{S(\Phi,l)}{S_o} \right)^{n-2} \int_{L^*} \int_{E_2} \sin^2(\xi, \Omega_{l\varphi}, \Phi) h(x, \xi) d\xi \ dx \ l \ d\Phi \ dl \ d\varphi \ d\omega \frac{d\varphi \ d\omega}{S_o^2 C(\omega) C(l, \varphi)}, \tag{16}\]

where \( l_0 \) and \( \phi_0 \) are arbitrarily small fixed numbers. From the regularity of the surface \( \partial B \) we obtain the Taylor expansion

\[
S(\Phi,l) = l S_0'(0,0) + \Phi S_{\Phi}'(0,0) + \frac{l^2}{2} S_{\Phi}'(0,0) + l \Phi S_{\Phi\Phi}'(0,0) + \frac{\Phi^2}{2} S_{\Phi\Phi}'(0,0) + R(\Phi,l), \tag{17}\]

where \( R(\Phi,l) = o(l^2 + \Phi^2) \). Here all functions continuously depend on \( l \) and \( \Phi \), as well as on \( \omega \) and \( \varphi \). Below, we will see, that \( S_0'(0,0) = S_{\Phi}'(0,0) = 0 \).

Using the mean value theorem we find

\[
\int_{L^*} \int_{E_2} \sin^2(\xi, \Omega_{l\varphi}, \Phi) h(x, \xi) d\xi \ dx = l^* \int_{E_2} \sin^2(\xi, \Omega_{l\varphi}, \Phi) h(x_0, \xi) d\xi, \tag{18}\]

where \( x_0 \) is a point from the segment \( L^* \) and \( l^* = |L^*| \).

After substitution (18) in (16) and a change of variables \( u = l \sqrt{n}, v = \Phi \sqrt{n} \) we get

\[
2\pi \mu([B]) = \lim_{n \to \infty} \left( \frac{n}{2} \right) \int_{(S^2)^2} \int_{0}^{2\pi} \int_{0}^{\phi_0} \left( 1 - \frac{S(\Phi,l)}{S_o} \right)^{n-2} \int_{E_2} \sin^2(\xi, \Omega_{l\varphi}, \Phi) h(x_0, \xi) d\xi \ dx \ l \ d\Phi \ dl \ d\varphi \ d\omega \frac{d\varphi \ d\omega}{S_o^2 C(\omega)}, \tag{19}\]

where \( l^* = l \cdot b(\omega, \varphi) + o(l) \).

One can interchange the limit and the integration operations. Substitution (17) in (19) and making use of

\[
\int_{E_2} \sin^2(\xi, \Omega_{l\varphi}, \Phi) h(x_0, \xi) d\xi \to \int_{E_2} \sin^2(\alpha(\xi, \omega, \varphi_1) h_{[P^*(\omega)]}(\xi) d\xi \tag{20}\]

almost everywhere when \( n \to \infty \), where \( \alpha(\xi, \omega, \varphi_1) \) is the angle between the direction \( \varphi_1 \) in the plane \( e_\omega \) and the intersection of \( e_\xi \) with the plane \( e_\omega \), \( [P^*(\omega)] \) is the
bundle of planes containing the point \(P^*(\omega) \in \partial B\) with normal \(\omega\), we obtain
\[
2\pi \mu([B]) = \int_{(S^2)} \int_0^{2\pi} \left[ \int_0^\infty \int_{-\infty}^{\infty} \exp \left( -\frac{u^2}{2} S''_l(0,0) - uv S''_q(0,0) - \frac{v^2}{2} S''_{q\Phi}(0,0) \right) \right] \times u^2 \, du \, dv \left[ \int_{\xi_1}^{\xi_2} \sin^2 \alpha(\xi, \omega, \varphi) h_{|P^*(\omega)|}(\xi) \, d\xi \right] \frac{b(\omega, \varphi)}{2C(\omega)} \, d\varphi \, d\omega, \tag{21}
\]
where \(b(\omega, \varphi) = \lim_{t \to 0} l_t^\gamma.\)

4. A Representation Obtained by Stochastic Approximation

It follows from (21) that the final representation for \(\mu([B])\) depends on values of \(S''_l(0,0), S''_q(0,0), S''_{q\Phi}(0,0)\) which are functions of \(\omega\) and \(\varphi\) (see (21)). It was proved in [7], that:
\[
S''_l(0,0), S''_q(0,0), S''_{q\Phi}(0,0) \text{ depend only on derivatives of at most order of two of the surface } \partial B \text{ at the point } P^* \text{ whose outer normal is } \omega.
\]
Hence the corresponding calculation we can do for the osculating paraboloid of \(\partial B\) at the point \(P^*(\omega)\) whose outer normal is \(\omega\). In [7] the following expressions for the derivatives in terms of the normal curvatures of \(\partial B\) at the point \(P^*(\omega)\) were found:
\[
\begin{align*}
S''_l(0,0) &= 0, \quad S''_q(0,0) = 0, \quad S''_{q\Phi}(0,0) = \frac{\pi \sqrt{k_1 k_2} r^2(\varphi)(k_3^2 \cos^2 \varphi + k_4^2 \sin^2 \varphi)}{2A^4}, \\
S''_{q\Phi}(0,0) &= \frac{\pi \sqrt{k_1 k_2} \sin 2\varphi(k_2 - k_1)}{2A^4}, \quad S''_{q\Phi}(0,0) = \frac{2\pi \sqrt{k_1 k_2} r(\varphi)}{A^2}, \tag{22}
\end{align*}
\]
where \(k_i, i = 1, 2\) are the main normal curvatures, \(r(\varphi) = k_1^{-1} \cos^2 \varphi + k_2^{-1} \sin^2 \varphi\) is the radius of the normal curvature in the direction \(\varphi\) at the point \(P^*(\omega)\) of \(\partial B\) and \(A = \sqrt{k_2^2 \cos^2 \varphi + k_3^2 \sin^2 \varphi}\). Also, in [7] was found that (see (21))
\[
b(\omega, \varphi) = \sqrt{\frac{\cos^2 \varphi}{k_1^2} + \frac{\sin^2 \varphi}{k_2^2}} \text{ and } \tan \varphi_1 = \tan \varphi \frac{k_1}{k_2}. \tag{23}
\]
Substituting (22) and (23) into (21) we get
\[
\mu([B]) = (2\pi^2)^{-1} \int_{(S^2)} \int_0^{2\pi} \left[ \int_{\xi_1}^{\xi_2} \sin^2 \alpha(\xi, \omega, \varphi) h_{|P^*(\omega)|}(\xi) \, d\xi \right] \frac{\sqrt{k_1 k_2}}{k^2(\omega, \varphi)} \, d\varphi \, d\omega, \tag{24}
\]
where \(k(\omega, \varphi)\) is the normal curvature in the direction \(\varphi\) at the point \(P^*(\omega)\) of \(\partial B\). Taking into account that
\[
\sin^2 \alpha(\xi, \omega, \varphi) = \cos^2 (\varphi - \psi), \tag{25}
\]
where \(\psi\) is the direction of the projection of \(\xi\) into the plane with normal \(\omega\), we get (3).

Theorem 1 is proved.

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References


