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A NUMBER THEORETIC IDENTITY

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In this note we discuss a certain number theoretic identity with some interesting connections. It is seen that not only can this identity be extended to more general situations, but it also has close connections with the notion of a congruent number. In fact the above-mentioned identity holds good in the case of square matrices as well as bounded linear operators. Beside some interesting geometric results emerge naturally as a consequence of this identity.

*Keywords***.** Number; theoretic; identity; congruent; triple; pythagoras.

1. Introduction

The number theoretic identity we are interested in is a generalization of the identity, $x^2 + (x+1)^2 + (x(x+1))^2 = [x(x+1)+1]^2$ (1)

If we expand both the sides separately then it is seen that the equality clearly holds. However what is interesting is the fact that the above equality arises as a special case of the identity,

$x^2 + (x+1)^2 + (p+m)^2 = (q+m)^2 \pm 2m$, where $p = x(x+1)$, $q = x(x+1) + 1$ and $m = 0, \pm 1, \pm 2, \ldots$

i) $x^2 + (x+1)^2 + (p+m)^2 = (q+m)^2 - 2m$, when $m = 0, 1, 2\ldots$

ii) $x^2 + (x+1)^2 + (p-m)^2 = (q-m)^2 + 2m$, when $m = 0, -1, -2\ldots$

It is seen that by simply setting $m = 0$, we get identity (1).

Now by expanding both the left hand side and the right hand side separately, the validity of this identity can be easily established for all $x \in R$.

For the case $m = 0$, we have a generalization of the identity (1). To this end we have the following theorem:

Theorem 1.1. *For x ∈ R,* $\sum_{k=0}^{2n+1}(x+k)^2$ *= 2[x(x+1)+.....+(x+2n)(x+2n+1)] + (n+1)*

for n= 0, 1, 2, 3.,......................

Proof: This can be established by appealing to the principle of mathematical induction.

2. Extension to the Matrix and the Bounded Linear Operator Cases

It is interesting to note that this identity holds good in the case of both square matrices as well as bounded linear operations on a linear space X. To this end we have the following proposition.

Proposition 2.1 *If A is a n x n square matrix or a bounded linear operator on a linear space X, then ,*

(I) $A^2 + (A+I)^2 + [A(A+I) + mI]^2 = [A(A+I) + I + mI]^2 - 2mI$,

(II) $A^2 + (A+I)^2 + [A(A+I) - mI]^2 = [A(A+I) + I - mI]^2 + 2mI$,

where I is the identity matrix or the identity linear operator, as the case may be, and m = 0, ± 1, ± 2, ...

Proof. The proof is along the same lines as in the case of the number theoretic identity. On expanding both sides of the equality separately we get the desired result.

It is worth noting that the above proposition is actually a consequence of a more general result stated below.

Theorem 2.2. *If A and B are two n x n matrices (or bounded linear operators on a linear space X), such that AB = BA and* $B^2 = B$*, than,*

$(AB)^2 + (AB+B)^2 + [A(A+B)]^2 = [A(A+B)+B]^2$.

Proof. The validity of this equality is established by expanding both the left hand side and the right hand side separately.

It is not difficult to see that by setting B = 1 in the theorem we recover proposition 2.1, for m = 0

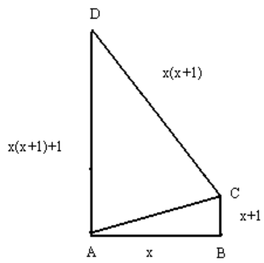
3. Some Geometry

Consider a right angled triangle ABC with the right angle subtended at the vertex B as shown in the figure below, with AB and BC having lengths x and x+1 respectively. Next we construct another angled triangle ACD with DC perpendicular to AC, and DC having length x(x+1). Now it follows from Pythagoras' theorem that,

$AB^2 + BC^2 + DC^2 = AD^2$

i.e, $x^2 + (x+1)^2 + [x(x+1)]^2 = AD^2$

It follows from the identity established earlier that $AD^2 = [x(x+1)+1]^2$.



Given this scenario we immediately see a close connection with the notion of a congruent number.

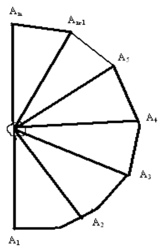
Definition 3.1. *A positive integer n is said to be a congruent number if there exist v, w, x, y, z rational numbers, such that* $v^2 + w^2 + x^2 + y^2 = z^2$ *and* . $\frac{vw}{2} + \frac{xy}{2} = n$.

If in the discussion preceding this definition x is an integer, then it follows from the equality $x^2 + (x+1)^2 + (\sqrt{(x^2+(x+1)^2)})^2 + \{x(x+1)\}^2 = [x(x+1)+1]^2 + [x^2+(x+1)]^2$, that

n = $\frac{x(x+1)}{2} + \frac{\sqrt{(x^2+(x+1)^2)}(x(x+1))}{2}$ is a congruent number so long as $(x^2 + (x+1)^2)$ is a perfect square. In this case it turns out that n is the sum of the areas of the two triangles.

Next, we prove the following theorem in an attempt to generalize the notion of Pythagorean Triple [2].

Theorem 3.2. *Consider the polygon shown below, comprising of n right angled triangles with every pair of adjacent triangles having a common side. We claim that* $OA_n^2 = OA_1^2 + A_1A_2^2 + A_2A_3^2 + ... + A_{n-1}A_n^2$



Proof. The proof follows from the Pythagoras theorem and the principle of mathematical induction.

4. Conclusion

In this paper we have discussed an interesting number theoretic identity, which has several generalizations and is closely connected with the notion of a congruent number. Here we have generalized this notion through a geometric approach.

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