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SOME APPLICATIONS OF EULERIAN GRAPHS

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The genius Swiss Mathematician Leonhard Euler who was a prolific contributor to several areas of Mathematics is considered as the inventor of the concept of a graph. Graphs have proved to be very useful in modeling a variety of real-life situations in many disciplines. This article is intended for the attention of young readers, uninitiated in graph theory and gives an introductory discussion of certain well-known application problems that involve graphs in their solution and in particular the role of Eulerian graphs starting from the problem of Konigsberg seven bridges to the current problem of DNA fragment assembly. We also point out connections of Eulerian circuits with drawing certain floor designs, known as "kolam".

Keywords: Graph; Eulerian circuit; Eulerian graph

1. Introduction

Topics under the broad title of "Discrete Mathematics" are intended to provide the mathematical foundation for pursuing courses relating to computer science besides giving a mathematical background for solving problems in many application areas. Graph theory is one such topic which has found use in a variety of application problems. The birth of Graph theory can be attributed to the work of the famous Swiss mathematician Leonhard Euler (1707 – 1783) in solving the problem of "Seven bridges of Konigsberg". We discuss this problem in a later section. The method used by Euler in solving this problem gave rise to the notion of 'graph', which essentially is a discrete structure useful for modelling relations among objects. The development of the subject of graph theory has therefore been phenomenal with the subject drawing from and contributing to many other disciplines of study. In this article we discuss the notions of a graph, Eulerian graph and certain application problems that involve Eulerian graphs, starting from the problem of Konigsberg seven bridges to the current problem of DNA fragment assembly, to draw the attention of the young researchers uninitiated in this area of study. We also bring out some connections of Eulerian circuits with certain floor designs, known as "kolam".

2. Acquaintance Problem and Graphs

A group leader of a company was addressing in a training programme, a group of 21 newly recruited trainees of the company, who have assembled for the programme. The leader who was fond of numbers was making a remark as follows: Among the 21 trainees assembled here it could be that there are trainees who know each other, trainees who know all others, trainees who do not know each other as well as trainees who do not know anyone in the group. But there will be a trainee in this group who already knows an even number (including the even number zero) of other trainees. The claim of the leader looks a little puzzling but it is true. What do we do to prove this claim? One way is to create a list, for each of the 21 persons, of other trainees in the group already known to them. The information obtained can be processed, although it could be a little cumbersome, in order to find out for each of the 21 trainees, the number of other trainees already known to them. This will reveal the truth of the claim.

On the other hand, it is interesting to note that this problem could be discussed with the use of graphs. We shall illustrate this with a smaller number of 7 trainees in the group, say Hussain, Johnson, Kamalesh, Khairul, Prasanna, Qadir, Ramesh. For convenience, we code their names as H,J,K1,K2,P,Q,R. Suppose H knows R (even before meeting R in this training programme); J knows K1,P,Q; K1 knows J,K2,Q; K2 knows K1; P knows J; Q knows J,K1; R knows H. It is clear that H knows only one other trainee; J knows three; K1 knows three; K2 knows 1; P knows 1; Q knows 2; R knows 1. We shall call these numbers as 'acquaintance numbers'. Thus the acquaintance numbers of H,J,K1,K2,P,Q,R are respectively 1,3,3,1,1,2,1. The claim of the leader is verified, since (at least one of them) Q knows an even number of other trainees.

The situation can be modeled by a 'graph'. For each trainee we associate a point (indicated by a small circle) in the plane. We label the points as H,J,K1,K2,P,Q,R. We join two points by a line (straight or curved) if the two trainees represented by the points 'know' each other. The resulting diagram which is an example graph, is in Figure 1. The number of lines at a point gives the acquaintance number of the corresponding person.

Consider another situation as follows: Suppose H knows only R; J knows only K1; K1 knows only J; K2 knows none; P knows only Q; Q knows only P; R knows only H. Then the corresponding graph is shown in Figure 2. The acquaintance numbers are 1,1,1,0,1,1,1. Still one of these numbers namely 0 is even, again verifying the claim. Note that if the number of trainees in our discussion is 6 or 8 instead of 7 , the claim need no longer be true. For example, if there are only 6 trainees H,J,K1,K2,P,Q such that H knows only J; J knows only H; K1 knows only K2; K2 knows only K1; P knows only Q; Q knows only P, then the acquaintance numbers are 1,1,1,1,1,1, all of which are odd.

So, for the claim to be true the number of trainees in the group should be odd. This fact can be proved as a consequence of a basic theorem, known as "Handshaking Theorem" in graph theory, which we shall discuss a little later.

Thus a graph is a discrete structure that gives a representation of a finite set of objects and certain relation among some (or all) objects in the set. We shall now express the notion of a graph and certain terms related to graphs in a little more rigorous way.

A graph G = (V,E) consists of a finite set V of elements, called vertices (also called points or nodes) and another finite set E of pairs of elements of V of the form (u,v) (also written as uv or vu). The element (u,v) is called an edge (also called line). In a diagrammatic representation, we denote a vertex by a small circle (or by a small dot) and an edge (u,v) by a line joining the points representing the vertices u and v. For example, in Figure 3, a graph is shown with 5 vertices u,v,w,x,y and 5 edges (u,x), (u,w), (x,w), (x,y), (v,y). Note that in a graph there could be more than one edge between two vertices which are referred to as multiple edges between the vertices or there could be an edge (called loop) joining a vertex with itself. Such a graph is called a multigraph. In Figure 4, a graph with multiple edges and loops is shown.

The number of edges at a vertex is called the degree of the vertex. The degree of the vertex u in the graph in Figure 3 is therefore 2. The degrees of v, w, x, y are respectively 1, 2, 3, 2. If we add all these degrees we have the sum = 2+1+2+3+2 = 10. This number is twice the number of edges, namely 5. In fact this is true in any graph and the result is referred to as “Handshaking Theorem”. Note that in the acquaintance graph in Figure 1, the number of edges is 6 and the sum of the degrees (or the acquaintance numbers) equals 1+3+3+1+1+2+1 = 12 and in Figure 2, the number of edges is 3 and the sum of the degrees equals 1+1+1+0+1+1+1 = 6.

Handshaking Theorem: In any graph, the sum of the degrees of the vertices is twice the number of edges.

If we denote the degree of a vertex v in a graph G = (V,E) by d(v) and if G has q edges, then this theorem can be expressed as follows:

$$\Sigma_{v \varepsilon V} d(v) = 2q.$$

Note that this theorem is a consequence of the fact that an edge (which is not a loop) is incident with two vertices, thus contributing degree one to each vertex. So while counting the degrees of vertices, such an edge will get counted twice, once at each end of the edge. (A loop is incident with a vertex two times and so contributes degree two to this vertex). In a graph a vertex with even degree is called an even vertex and a vertex with odd degree as an odd vertex. The sum of the degrees of even vertices is an even number since we always have an even integer when we add any number of even integers. But only the sum of an even number of odd integers is an even integer. As a consequence of Handshaking Theorem, the number of odd vertices in a graph has to be even. In fact in a graph with an odd number of vertices, the vertices of odd degree will be even in number and so at least one vertex must have even degree. This is the reason why the claim of the group leader in the acquaintance problem with an odd number of trainees that there is at least one trainee who knows an even number of other trainees, is true. For example, in the graph with seven vertices in Figure 1, there are an even number (six) of odd vertices that correspond to trainees who know an odd number of other trainees.

3. Diagram Tracing and Eulerian graph:

A popular old game that entertains children runs as follows: Can you trace with a pencil a diagram of points (with the small circles representing the points) and lines as shown in Figures 5a and 5b? The condition is that the diagram is to be traced beginning at a point and on completion end at the same point but the pencil should not be lifted till the diagram is completely traced and a line in the diagram should not be retraced (i.e. can be traced only once). A curious child will certainly try to find the answer by trial and error and will arrive at the conclusion after sometime that it is not possible in diagram in Figure 5b but it is possible in the diagram in Figure 5a. The question of whether it is possible to trace such a diagram, can be quickly answered if the concept of Eulerian graph is known. Indeed the diagram in Figure 5a is Eulerian whereas the diagram in Figure 5b is not.

We now introduce the concepts of path and circuit in a graph to enable us to describe the notion of an Eulerian graph in a little more rigorous way.

If e = xy is an edge in a graph, then x is called the start vertex and y, the end vertex of e. A path P (or u-v path P) in a graph is a sequence of edges so that the end vertex of an edge in the sequence is the start vertex of the next edge in the sequence and the path begins in the vertex u and ends in the vertex v. If vertices u and v are the same, then the path is called a circuit (some call it cycle). For example in the graph in Figure 6, (a,b)(b,c)(c,b)(b,c)(c,d) is a path and (a,b)(b,c)(c,b)(b,c)(c,a) is a circuit. Vertices and/or edges can be repeated in a path or in a circuit. (A path is called a walk by some authors. Due to the diversity of people who use graphs for their own purpose, the naming of certain concepts has not been uniform in graph theory).

An Eulerian path in a graph is a path which uses all the edges of the graph but uses each edge exactly once. An Eulerian circuit is a circuit which has a similar property. Note that in an Eulerian path or an Eulerian circuit, a vertex can be visited more than once but not an edge ! An Eulerian circuit begins and ends at the same vertex. A graph that has an Euler circuit is known as an Eulerian graph. For example in the graph in Figure 6, (a,b)(b,c)(c,e)(e,d)(d,c)(c,a) is an Eulerian circuit and hence the graph in Figure 6 is an Eulerian graph.

4. The Problem of Seven Bridges

The year 1736 when Euler solved the problem of seven bridges of Königsberg is taken to mark the birth of graph theory. The seven bridges problem is a well known problem that can be stated as follows: The Pregel river in the town of Königsberg divided it four land regions with a central island. These four land regions were connected by seven bridges as shown in the diagram in Figure 7a where the land areas are denoted by the letters A,B,C,D. It is said that the people of the town used to ponder over the question of whether it is possible to start in a land area and make a walk through the bridges visiting each bridge exactly once and returning to the starting point. In solving this problem of “Seven bridges”, Euler developed an abstract approach which is considered to be related to the “geometry of position”. Represented in terms of a graph E_g (Figure 7b), the four land areas A, B, C, D are the vertices and the seven bridges are the edges of the graph. The vertices A and B are joined by multiple edges (two edges), so that the graph E_g is a multigraph! The “Seven bridges problem” can be expressed in graph-theoretic terms as follows: Is there an Eulerian circuit in the graph E_g? The answer from Euler’s 1736 paper to this question is NO! . This is stated as an important theorem in the study of Eulerian graphs.

Theorem on Eulerian graphs: A connected graph with two or more vertices is an Eulerian graph (ie. has an Eulerian circuit) if and only if each vertex of the graph has even degree.

Note that the necessary part of the theorem is based on the fact that, in an Eulerian graph, everytime a circuit enters a vertex through an edge it exits the vertex through another edge, thus accounting for an even degree two at the vertex. At the starting vertex, the circuit initially exits and finally enters when the circuit is completed.

In the graph E_g in the Seven bridges problem, all the four vertices have odd degree. So it cannot have an Eulerian circuit.

5. Chinese Postman Problem

All vertices of a graph need not be of even degree and so a graph may not be Eulerian. But if a graph has all except two vertices of even degree then it has an Eulerian path which starts at one of the odd vertices and ends at the other odd vertex. A graph having an Eulerian path but not an Eulerian circuit is called semi-Eulerian. For example in the graph in Figure 8, (a,b)(b,c)(c,d)(d,b)(b,e)(e,d)(d,f) is an Eulerian path and hence the graph in Figure 8 is semi-Eulerian.

In a graph G_p that models streets and street corners in a town, with the street corners as the vertices and the streets as the edges, some one starting in a street corner and walking along the streets one after another can end up in a street corner, giving rise to a path in the graph. Note that walking along a street more than once from one corner to another, corresponds to repeating an edge or a vertex in the graph.

Suppose that a postman has to deliver letters to the residents in all the streets of a village. Assume that the village is small enough for the postman to be assigned this task everyday. If the graph G_p that represents the streets and street corners as mentioned above, is semi-Eulerian so that there is an Euler path in the graph G_p, then this path gives rise to a route following which the postman can start in a street corner, deliver letters going through every street exactly once. This is a desirable situation for the postman. But if no such Euler path exists in G_p, then the postman may have to repeat some of the streets. This problem is called the Chinese postman problem in honour of a Chinese mathematician Meigu Guan who proposed this problem.

6. DNA fragment assembly

DNA (deoxyribonucleic acid) is found in every living organism and is a storage medium for genetic information. A DNA strand is composed of bases which are denoted by A (adenine), C (cytosine), G (Guanine) and T (thymine). The familiar DNA double helix arises by the bondage of two separate strands with the Watson-Crick complementarity (A and T are complementary; C and G are complementary) leading to the formation of such double strands.

DNA sequencing and fragment assembly is the problem of reconstructing full strands of DNA based on the pieces of data recorded. It is of interest to note that ideas from graph theory, especially Eulerian circuits have been used in a recently proposed approach to the problem of DNA fragment assembly. We do not enter into the details but only mention that this brings out the application of graph theory in the field of bioinformatics. More elaborate details on this topic can be found in Kaptcianos (2008) and Pevzner et al (2001).

7. Floor Designs and Eulerian graphs

Traditional interesting floor designs, known as “kolam”, are drawn as decorations in the floor and in large sizes and in interesting shapes, during festivals and weddings, with the drawing done with rice flour or rice paste especially in South India. See for example Nagata and Robinson (2006), Siromoney (1978) and Siromoney et al (1974). Generally, in drawing a kolam, first a suitable arrangement of dots is made and then lines going around the dots are drawn. An example kolam is shown in Figure 9 where the dots are ignored. One of the types of kolam, is known as ‘kambi kolam’ (the Tamil word kambi meaning wire). In this type, the kolam is made of one or more of such ‘kambi’s . The kolam in Figure 9 is of this type and is made of a single ‘kambi’ whereas the kolam in Figure 11 is made of three ‘kambi’s.

A kolam drawing can be treated as a special kind of a graph with the crossings considered as vertices and the parts of the kambi between vertices treated as edges. The only restriction is that unlike in a graph, these edges can not be freely drawn as there is a specific way of drawing the kolam. The single kambi kolam will then be an Eulerian graph with the drawing starting and ending in the same vertex and passing through every edge of the graph only once. In Figure 10, the kambi kolam of Figure 9 is shown as a graph with vertices (indicated by small thick dots) and edges and this graph is Eulerian as every vertex is of degree 4. Note that the graph of kambi kolam (with more than one kambi) in Figure 11 will also be Eulerian but the drawing of the kolam in the way it will be done by the kolam practitioners will not be giving rise to the Eulerian circuit. On the other hand in the case of single kambi kolam it is of interest to note that the drawing of the kolam will give a tracing of an Eulerian circuit in the corresponding graph.

8. Conclusion

With an intention to bring to the notice of young readers, some of the applications of graphs, especially Eulerian graphs are pointed out in this article. There are a number of other interesting applications where graphs have found their use. An account of some of the other applications is given in Ismail (2009). The authors acknowledge consulting the books (Akerkar and Akerkar (2004), Arumugam and Ramachandran (2001), Fleischner (1990), Rosen (2007)) in the preparation of this article. The interested reader is encouraged to refer to them and the references in them for more details.

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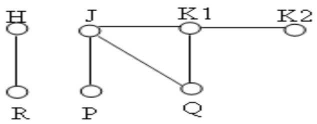

Fig. 1 Graph of an acquaintance of trainees

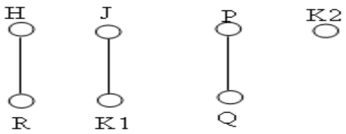

Fig. 2 Graph of another acquaintance of trainees

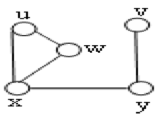

Fig. 3: A graph on five vertices

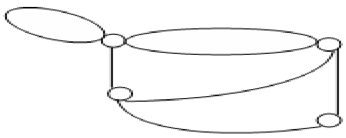

Fig.4 : A multigraph

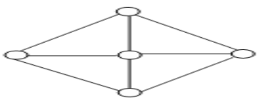

Fig. 5b: Diagram that cannot be traced

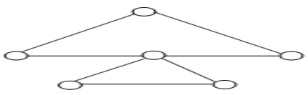

Fig. 5a : Diagram that can be traced

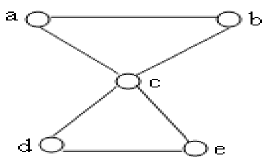

Fig. 6: An Eulerian graph

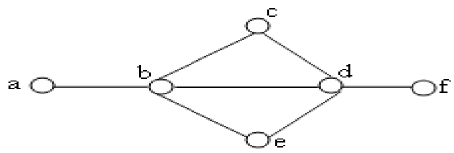

Fig. 8: A semi-Eulerian graph

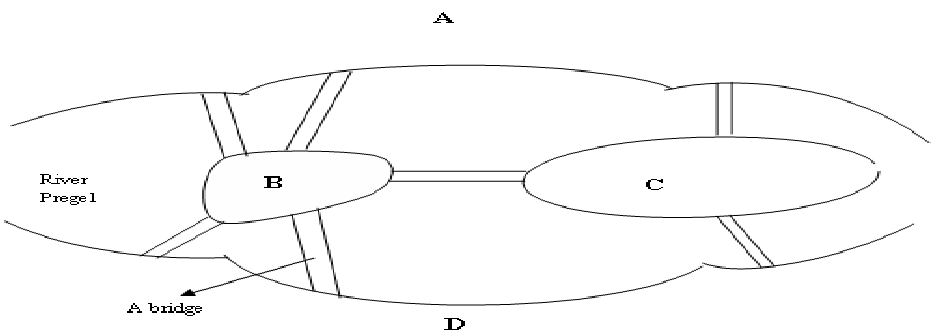

Fig. 7a: Seven bridges of Konigsberg - a sketch

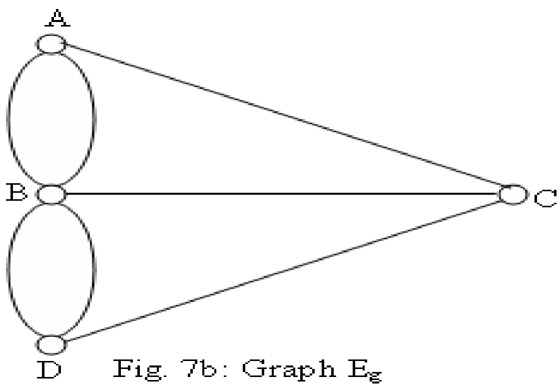

Fig. 7b: Graph E_g

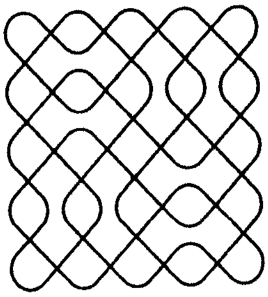

Fig. 9: A single kambi kolam

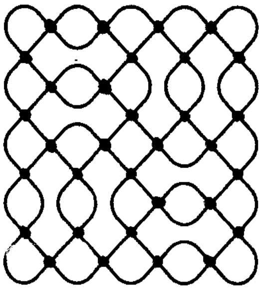

Fig.10: Graph of Kolam in Fig.9

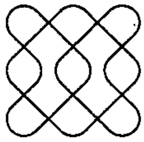

Fig. 11: A kolam made of three Kambis

Brief Biographical sketch of the authors:

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K.G. Subramanian is a visiting Professor from 2007 at the School of Mathematical Sciences, Universiti Sains Malaysia, after retiring in 2006 from the Department of Mathematics, Madras Christian College, Chennai, India. His main area of research is two-dimensional picture array languages. His PhD supervisor is the eminent Indian formal language-theorist Prof. Rani Siromoney.