

DECEPTIVE PATTERNS IN MATHEMATICS

[2nd title: Deceptive and Misunderstanding Nature of Patterns in Mathematics]

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Mathematics is sometimes described as a study of patterns. Patterns in mathematics can be deceptive and therefore can lead to incorrect conclusion regarding the general term. Similarly patterns may not reveal correctly the nature of their members. This comes in the way of proper understanding of the mathematical concepts, like sequence. With some concrete examples we throw, in this expository article, searching light on the patterns and their deceptive nature and help the reader to consolidate the correct understanding and appreciation of the term 'sequence'.

Keywords: sequence, generating-function, primes, circles, Venn-diagrams.

1. INTRODUCTION

Nature is full of patterns. Some patterns are pleasing, while some others are challenging. Some people define mathematics as 'study of patterns.' By observing part of a pattern, can we recognize or identify the remaining part of that pattern? Many patterns deceive us, and we get surprised by the unexpected turn that the pattern takes in the part that follows the part of that pattern that was seen. Especially in mathematics it is all the more challenging to find the general form, of any pattern, that will unfold or generate all the observed individual members of the pattern. In this article, we shall study and examine some patterns of such type.

2. NEXT MEMBER IN A PATTERN

Consider the pattern whose initial terms are

$$2, 4, 6, 8, \dots \quad - (1)$$

What is the next term in this pattern? This type of question in which some initial terms are given and we are asked to recognize the next term, is very common in many examinations meant not only for mathematics students, but also in examinations designed to select even clerks. Tests in which such questions are set are many times called 'intelligence tests', or 'psychology tests', or 'common-sense tests'. Readers will draw their own conclusion, after reading this article, as to how far these adjectives used for such tests are right.

What is the answer to the question which asks the next term in (1)? Most people would say 10. To see whether this answer is correct or not, we must examine the reasoning that would be put forward by those who give the answer 10. Different persons giving the same answer 10,

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could give the reasoning in different ways. Some would say, “In (1) we record the number 2, drop the next number 3, then record the next number 4, again ignore the next number 5, record 6, omit 7, and record 8. So we must follow this pattern, discard the next number 9, and record 10. Hence the answer is 10.” Some others would say, “Starting with 2, we are noting the alternate integers. Therefore the answer is 10.” Somebody else may argue, “We are writing even numbers starting with 2. The even number after 8 is 10. So the required answer is 10.” All these arguments convey the same reasoning even though they are worded differently. Some other mathematically inclined person may state the same reasoning as follows: “Numbers given in (1) are generated by the formula $2n$. If we substitute $n = 1, 2, 3, 4$ in $2n$ we get 2, 4, 6, 8 respectively. So if we put the next value of n , namely 5, in $2n$, we get 10. Hence 10 is the expected answer. Thus $2n$ can be taken as the generating function of the pattern given by (1).”

Now consider the formula

$$n^4 - 10n^3 + 35n^2 - 48n + 24 \quad \text{-- (2)}$$

If we put $n = 1, 2, 3, 4$ in (2) we get 2, 4, 6, 8 respectively. So why not consider (2) as the generating function of (1). In that case, if we put $n = 5$ in (2) we get 34. So 34 can be taken as the answer to the query “what is the next term in (1)?”

One may be surprised with this and may wonder how one discovered the formula (2). It is easy to see that we want a formula (or expression) in n that will generate the same numbers (viz., 2, 4, 6, 8) that are generated by the formula $2n$, when we substitute $n = 1, 2, 3, 4$ in it. So to get such a formula we must add to (or subtract from) $2n$ the term(s) that would become zero when $n = 1, 2, 3, 4$. So

$$2n + (n - 1)(n - 2)(n - 3)(n - 4) \quad \text{-- (3)}$$

will be that formula. If we simplify (3) we get (2). Not only this, but we can construct infinite number of such formulae, that will generate 2, 4, 6, 8 when we put $n = 1, 2, 3, 4$ respectively in each one of them. To see this, look at

$$2n + a(n - 1)(n - 2)(n - 3)(n - 4) \quad \text{-- (4)}$$

where a can be any one of the values 2, 3, 4, 5, (in fact, any number for that matter), and it is easy to see that in (4) we have got that infinite number of formulae, since for any such value of a we get 2, 4, 6, 8 when we put $n = 1, 2, 3, 4$ respectively in (4).

The moral of all the discussion above is that there cannot be a unique answer to the question posed at the beginning of this section, and so, such question should not be asked, really speaking. Many, not so enlightened by the mathematical precision of thinking, would call (1) a sequence. In the study of Mathematics ‘sequence’ is a very important and basic concept, and what is very important to know, understand, and remember (as has been very clear from our example under discussion) is that

A SEQUENCE CANNOT BE DEFINED BY ENUMERATING FINITE NUMBER OF TERMS EVEN THOUGH THEY ARE LARGE IN NUMBER.

In other words, SEQUENCE DOES NOT GET DEFINED UNLESS ITS GENERAL TERM IS STATED.

--- (5)

In fact, the fact (5) is very much embodied in the definition of the term ‘sequence’ which states that ‘sequence is a function whose domain is the set of natural numbers.’ But it is rarely properly understood, as is clear from the example we have discussed just now. Thus

$$2, 4, 6, 8, \dots, 2n, \dots \quad \text{--- (6)}$$

is a sequence. Similarly,

$$2, 4, 6, 8, \dots, (n^4 - 10n^3 + 35n^2 - 48n + 24), \dots \quad \text{-- (7)}$$

is also a sequence which is different from the one in (6). But (1) is not a sequence.

One can easily see that even if we enumerate large number of terms, finite in number, in (1), we can always construct many general formulae, like the ones in (3) or (4) each of which will generate all those large number of terms.

Thus mere enumeration of a finite number of terms (without giving the general term), one followed by another, forming a pattern could deceive us, as is done by (1), if we try to find a general term that generates all those given finite number of terms. One has to be careful in such tasks.

One can see a simile here. We have, on our earth, billions of human beings, each having one nose, one mouth, two ears, and two eyes; but each one is different, and can be identified, distinguished, and recognized as different. Similarly we have billions of sequences each having its first four terms as 2, 4, 6, 8. In order to recognize, distinguish, and identify each one of them, we have to specify the general term like $2n$, or $n^4 - 10n^3 + 35n^2 - 48n + 24$, or the like. In fact, unless such a general term is specified, it is not eligible to be called or named as a sequence in Mathematics. We have not to ignore and forget this lesson brought forward by all the discussion of this section.

3. FERMAT GETS DECEIVED

Many great mathematicians have also been deceived, by enumeration of finite number of terms, in recognizing and identifying the pattern exhibited by them. For example, Fermat (1601 – 1665), a great French Mathematician saw the following five numbers in that order:

$$3, 5, 17, 257, 65537 \quad \text{-- (8)}$$

Fermat noticed that these are prime numbers and can be generated by the formula

$$\{2^{(2^n)}\} + 1 \quad \text{-- (9)}$$

when we substitute $n = 0, 1, 2, 3$, and 4 respectively, in it. So Fermat contemplated and thought for a while and declared that (9) would serve as a generating function for primes. He must have been delighted to feel that he discovered, by examining the pattern exhibited by (8), a formula that generates primes, an important achievement in the study of primes. But alas! His examination of (8) deceived him, This became evident when another great mathematician Euler (1707 – 1783) pointed out, in 1737 [i.e. about 100 years after Fermat’s examination of the pattern (8)] that if we put $n = 5$ in (9), we get

$$\{2^{(2^5)}\} + 1 = 641 \times 6700417 \quad \text{-- (10)}$$

which, therefore, is not a prime. In fact, no integer $n (> 5)$ has yet been found such that $\{2^{(2^n)}\} + 1$ is a prime. So one learns an exceedingly important lesson that a pattern displayed by enumeration of finite number of terms can turn out to be deceptive in conveying its nature and hence it is absolutely necessary to be extra careful, cautious, and vigilant while examining patterns in mathematics.

4. MISTAKEN JUDGEMENT OF POLIGNAC

We shall cite one more example of deceit, of one more mathematician, by the pattern exhibited by the following enumeration of the finite number of terms:

$$3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57 \quad \text{--- (11)}$$

Polignac, a French mathematician, in the year near about 1850, observed the following pattern in (11):

$$3 = 2^0 + 2; \quad 5 = 2^1 + 3; \quad 7 = 2^1 + 5 = 2^2 + 3; \quad 9 = 2^1 + 7 = 2^2 + 5; \quad 11 = 2^2 + 7 = 2^3 + 3; \\ 13 = 2^1 + 11 = 2^3 + 5; \quad 15 = 2^1 + 13 = 2^2 + 11 = 2^3 + 7; \quad 17 = 2^2 + 13; \quad 19 = 2^3 + 11 = 2^4 + 3;$$

$$21 = 2^1 + 19 = 2^2 + 17 = 2^3 + 13 = 2^4 + 5; \quad 23 = 2^2 + 19 = 2^4 + 7; \quad \text{etc. etc. --- (12)}$$

From this observation, he concluded that

$$\text{EVERY ODD POSITIVE INTEGER CAN BE EXPRESSED AS A} \\ \text{SUM A PERFECT POWER OF TWO AND A PRIME NUMBER.} \quad \text{--- (13)}$$

Here again Polignac was deceived in his process of observation, though he had actually claimed that he had examined the odd integers up to ten million. Later a critical eye of another investigator noticed that the property exhibited in (12) is not possessed by every odd positive integer, as claimed by Polignac, since he could discover a counterexample, namely 127. We can convince ourselves about this fact by noticing that

$$127 = 2^0 + (2 \times 63) = 2^1 + (5 \times 25) = 2^2 + (3 \times 41) = 2^3 + (7 \times 17) \\ = 2^4 + (3 \times 37) = 2^5 + (5 \times 19) = 2^6 + (7 \times 9) \quad \text{--- (14)}$$

and noting that $2^7 = 128 > 127$ so that only powers of 2 that need be taken into consideration for this verification are 0 through 7.

Thus Polignac was not cautious and attentive enough in examining (11) before formulating and consolidating his conclusion, or in other words, one can say that patterns in (11) and (12) deceived him to come to a hasty conclusion (13).

5. PARTITION OF CIRCLE'S INTERIOR

Let us now take an illustration from Geometry. Consider a circle. Take n points on its circumference. Join every pair of points so chosen by a line segment. Each such line segment will be a chord of that circle. The points on the circle should be so taken that the chords when drawn as described above would divide the interior of the circle in maximum possible number of compartments (i.e. parts). Count the maximum number of parts into which the circle can be partitioned by such a construction. Look at the Figures 1(a) through 1(e) in which $n = 1, 2, 3, 4, 5$ respectively. The maximum number of parts in which a circle can be divided by such construction is noted in Table 1.

n	1	2	3	4	5	
TABLE 1: -----						
Maxi number of parts into which circle is partitioned	1	2	4	8	16	-- (15)

So what would be the maximum number of parts into which the interior of a circle will be partitioned if we take $n = 6$ points on it and carry out the construction described above? Members of the pattern exhibited in (15) can be generated by the formula

$$2^{(n-1)} \quad \text{-- (16)}$$

So one would like to conclude that maximum number of compartments in which a circle is partitioned by the chords drawn by joining every pair of points selected from the n points situated on the circle is given by the formula (16) and hence the number of such compartments will be $2^{(6-1)} = 2^5 = 32$ when we take 6 points on a circle and draw the chords as stated above.

But if we actually draw such a figure, as is done in Figure 2, we realize that one cannot get more than 31 compartments when $n = 6$. In fact, (16) is not the correct formula in this situation and we have been deceived, in our examination, by the pattern displayed in (15). One can prove mathematically that the members in the pattern (15) can also be generated by the formula

$$(1/24)(n^4 - 6n^3 + 23n^2 - 18n + 24) \quad \text{-- (17)}$$

and that (17) really gives the correct formula for the problem on hand here. Hence the maximum number of compartments in which a circle can be partitioned is 31 when 6 points are taken on the circle and the construction of drawing chords is carried out as described above. Is this not a glaring case, of deception by a pattern, warning us to be extremely cautious and careful in dealing with patterns, however simple they may look?

6. CIRCLES AND VENN-DIAGRAMS

We shall study one more concrete illustration related to drawing of figures. Venn diagrams are employed to pictorially represent sets. We generally use different circles to represent different individual sets while drawing Venn diagrams. See Figures 3, 4, 5 in which 1, 2, and 3 sets (named using letters A, B, C) are shown respectively (using circles) as subsets of a universal set U represented by a rectangle following usual convention and practice. We notice that

UNIVERSAL SET IS PARTITIONED, AS WOULD BE EXPECTED, INTO $2^1 = 2$, $2^2 = 4$, and $2^3 = 8$ PARTS (SUBSETS) IN Figures 3, 4, and 5 RESPECTIVELY BY THE CIRCLES IN THEM and these numbers display a certain pattern. -- (18)

When we have to deal with 4 sets, we know that the maximum number of subsets into which the universal set should get partitioned is $16 = 2^4$. From the pattern observed in Figures 3, 4, 5 we expect that when 4 different circles would represent 4 different sets intersecting each other, the maximum number of compartments into which the universal set would get partitioned must be $2^4 = 16$. But again this turns out to be an eye-opening case of deception carried out by the patterns displayed by the Figures 3, 4, 5. Look at the Figure 6, and learn from it the fact

YOU MAY DRAW 4 INTERSECTING CIRCLES, REPRESENTING 4 SETS, ANY WAY YOU PLEASE, BUT IN THAT CASE THE MAXIMUM NUMBER OF COMPARTMENTS INTO WHICH THE UNIVERSAL SET WOULD BE PARTITIONED CANNOT EXCEED 14. -- (19)

This deception teaches us an important lesson, namely,

WHEN WE WANT TO DEAL WITH MORE THAN 3 SETS, CIRCLES ALONE SHOULD NOT BE EMPLOYED TO REPRESENT THE VARIOUS SETS WHILE DRAWING VENN DIAGRAMS. -- (20)

Otherwise the practical situation represented by 4 or more sets will not get correctly reflected in pictorial representation carried out by Venn diagrams. Thus

WHEN WE DEAL WITH 4 OR MORE SETS, WE ARE PER-FORCE REQUIRED TO EMPLOY DIFFERENT FIGURES LIKE RECTANGLES, OVALS, CIRCLES, ETC. TO REPRESENT DIFFERENT SETS WHILE USING VENN-DIGRAM REPRESENTATION in order to get real situation correctly and properly reflected in the pictorial representation. — (21)

7. CONCLUSION

Thus, in conclusion, we state that though Mathematics, in some sense, can be considered as a study of patterns, one must be aware of deceptive nature of patterns and should be extremely cautious and careful in studying, observing, distinguishing, and recognizing patterns, and especially in identifying the general term from the given or known finite number of terms of any pattern that is observed or noticed by us.. It is important that we must not accept any formula as a general term of a given pattern representing a certain practical situation unless it is mathematically proved to be so.

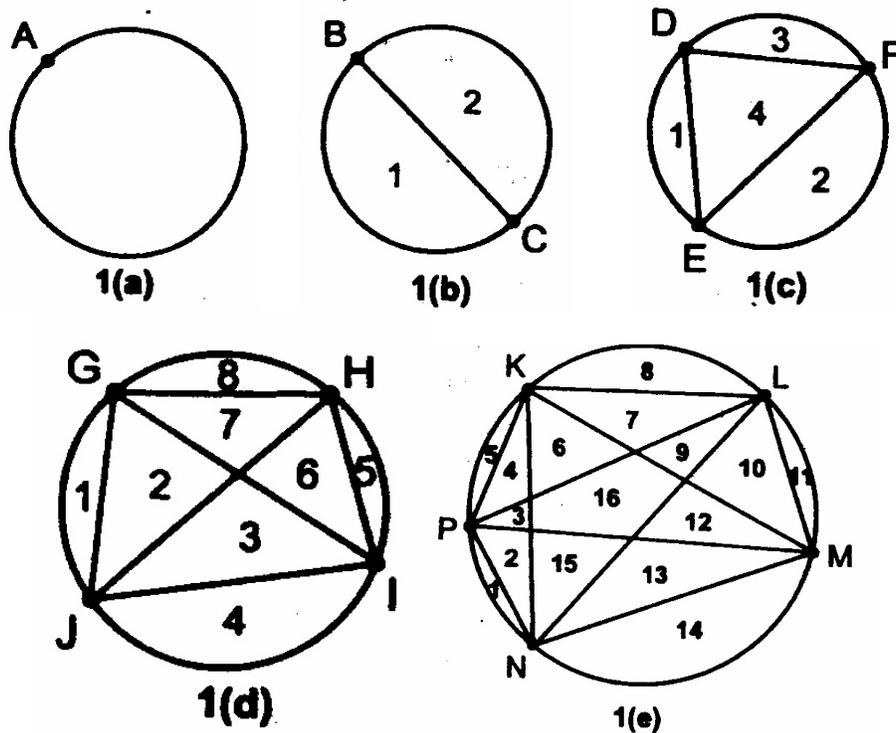


Figure 1

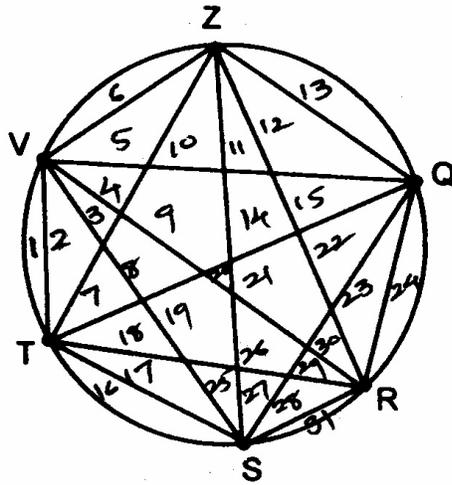


Figure 2

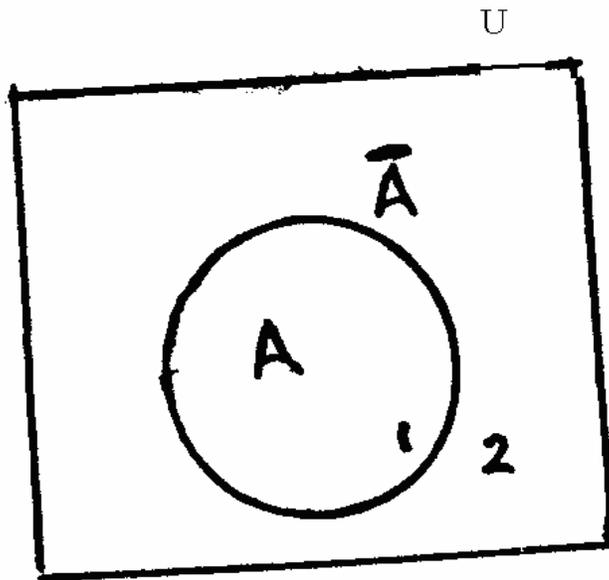


Figure 3

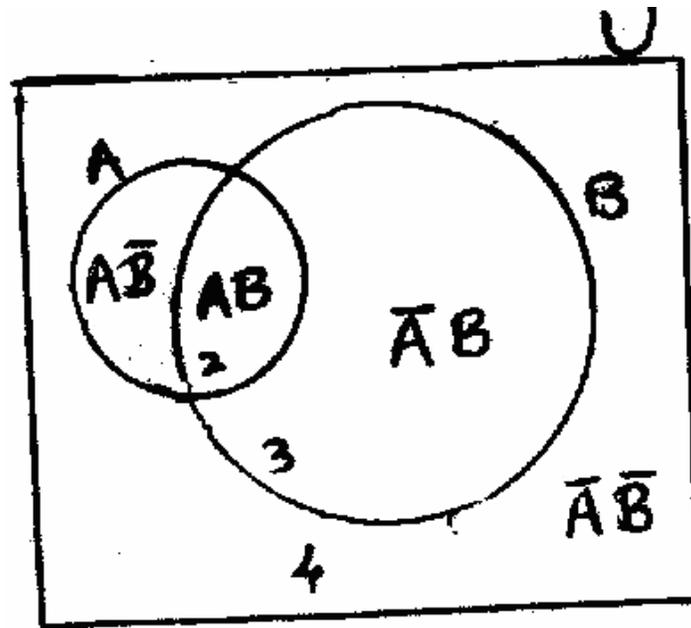


Figure 4

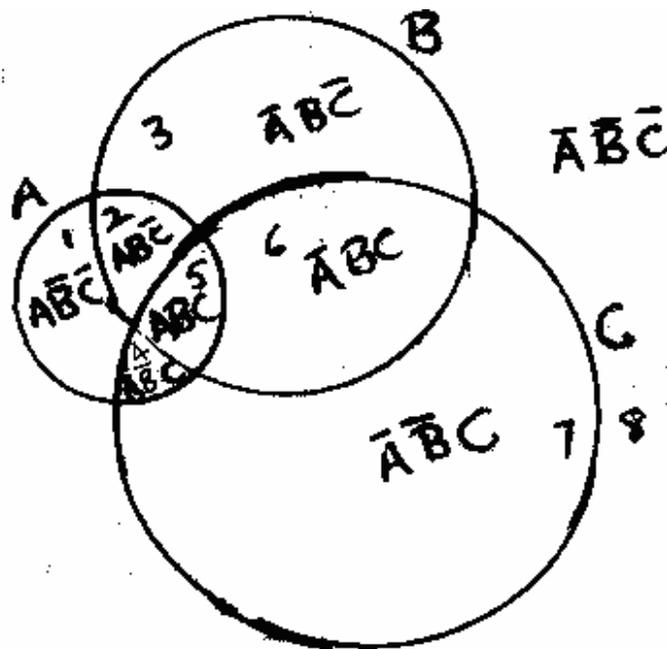


Figure 5

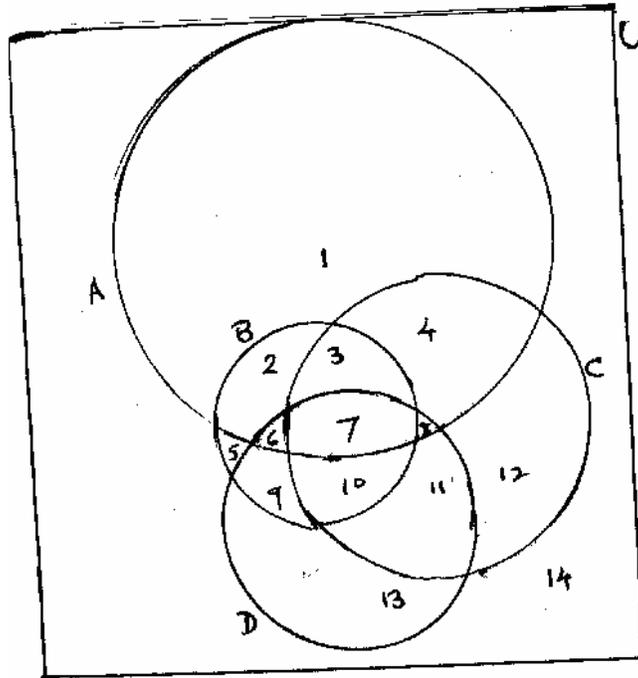


Figure 6

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