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TOMOGRAPHY OF BOUNDED CONVEX DOMAINS

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The present paper is a review of results in tomography of bounded convex domains. In the last century German mathematician W. Blaschke formulated the problem of investigation of bounded convex domains in the plane using probabilistic methods. In particular, the problem of recognition bounded convex domains $\mathbb{D}$ by chord length distribution function (or density function). Till recently explicit expressions for the chord length distribution functions have been known in the case when $\mathbb{D}$ is a disc, a regular triangle (see R. Sulanke, 1961) and a rectangle (see W. Gille, 1988). Using delta-formalism in Pleijel identities we have obtained explicit expression of the chord length distribution function for any regular polygon (see V. K. Ohanyan and H. Harutyunyan, 2009). In the last years the notion of orientation-dependent chord length distribution function have been introduced. In the paper of N. Aharonyan (2008) an explicit formula for orientation-dependent chord length distribution function for any bounded convex domains $\mathbb{D}$ have been obtained. These questions are connected with Covariogram Problem: Does the covariogram determine a convex domain, among all convex domains, up to translations and reflections? G. Matheron in 1986 conjectured a positive answer for this problem. In fact, the covariogram problem is equivalent to the problem of determining a convex domain from all its orientation-dependent chord length distributions (see W. Nagel, 1993; G. Averkov and G. Bianchi, 2007). All these problems are the problems of geometric tomography (see R. Gardner, 1995), since orientation-dependent chord length distribution function is the probability that parallel X-ray in direction ϕ less than or equal to y.

1. Description of results

Complicated geometrical patterns occur in many areas of science. Their analysis requires creation of mathematical models and development of special mathematical tools.

The corresponding area of mathematical research is called Stochastic Geometry (see [1], [2] and [10]). Among more popular applications are Stereology and Tomography (see [14], [3], [16], [21]).

The methods of form analysis are based on analysis of the objects as figures, i.e. as subsets of the plane. For these sets, geometrical characteristics are considered that are independent of the position and orientation of the figures (hence they coincide for congruent figures). Classical examples are area and perimeter of a figure. In the late thirties of the last century German mathematician W. Blaschke formulated the problem of investigation of bounded convex domains in the plane using probabilistic methods. In particular, the problem of recognition bounded convex domains $\mathbb{D}$ by chord length distribution function. Random lines generate chords of random length in convex domain $\mathbb{D}$. The corresponding distribution (or density) function is called the chord length distribution function that we denote by $F(y)$ (or chord length density function $f(y)$). In the initial stage of investigation mathematicians tried to find explicit expressions of the chord length distribution (or density) functions for concrete domains $\mathbb{D}$ in the terms of elementary functions. Till recently explicit expressions for the chord length distribution functions have been known in the case when $\mathbb{D}$ is a disc, a regular triangle (see [6]) and a rectangle (see [7]). These results have been obtained using the definition of chord length distribution function for a domain $\mathbb{D}$ (i.e. definition of the geometric probability in the space of lines in the plane). A family of identities primarily associated with isoperimetric inequalities for planar convex domains was discovered by A. Pleijel (see [4] and [5]) in 1956. Using combinatorial principles in integral geometry (in the meaning of W. Blaschke) R. V. Ambartzumian proved these identities (see [1], [2]) and pointed out (see [2], section 6.10) that classical Pleijel identities can be used for the finding chord length distribution function for the polygons which have no parallel sides. In the recent years our group has obtained important results in this direction. Using delta-formalism in Pleijel identities we have obtained explicit expression for the chord length distribution function of a regular polygon (see [22]). In the particular cases of a regular triangle, a square, a regular pentagon and a regular hexagon our formula for the chord length distribution function coincides with formulas available in the literature (see [6], [7], [8] and [9]) for $n = 3$, 4, 5 and 6 correspondingly. The form of the length density function is related to certain features of the corresponding figures. For example, poles of this function are related to parallel pieces of the contour and the form of $f(y)$ for y close to its maximum is essentially related to smaller details of the contour. The question is related to some methods of pattern recognition (see [17], [18] and [23]). The authors of [19] have shown that chord length distribution function does not characterize the convex domain $\mathbb{D}$. In [19] an example of two noncongruent convex domains with the same distribution function of the chord length have been constructed. The determination of the chord length distribution function has a long tradition of application to collections of bounded convex bodies forming structures in metal and ceramics. The series of formulae for chord length distribution functions may be of use in finding suitable models when empirical distribution functions are given (see [10], page 116). The results concerning certain infinite cylinders with regular polygonal bases are obtained in [11] and [15].

In the last years have been introduced the notion of orientation-dependent chord length distribution function $F_\phi(y)$, while F(y) is called mixed orientation distribution function. Using delta-formalism we have obtained orientation-dependent chord length distribution function (disintegration of mixed oriented chord length distribution function) (see [12]). These questions are connected with Covariogram Problem: Does the covariogram determine a convex domain, among all convex domains, up to translations and reflections? G. Matheron in 1986 conjectured a positive answer for this problem (see [24]). In fact, the covariogram problem is equivalent to the problem of determining a convex domain from all its orientation-dependent chord length distributions (see [13] and [20]).

At the Conference on Tomography at Oberwolfach in 1990, R. Gardner introduced the term geometric tomography. In the R. Gardner book [21], the following definition is offered: “Geometric tomography is the area of mathematics dealing with the retrieval of information about a geometric object from data about its sections, or projections, or both”. The word projection is used in the sense of a shadow, that is, the usual orthogonal projection on a line. The parallel X-ray of $\mathbb{D}$ in the direction ϕ gives for each $x \in \phi^{\perp}$ the length of the chord of intersection of $\mathbb{D}$ with the line through x parallel to ϕ. A set of four directions whose slopes have a transcendental cross ratio will ensure that the corresponding parallel X-rays determine each planar convex domain (see [21]).

2 Chord length distribution functions

Let $\mathbb{G}$ be the space of all lines g in the Euclidean plane $\mathbb{R}^2$, $(p, \varphi) =$ the polar coordinates of the foot of the perpendicular to g from the origin O, be standard coordinates for a line $g \in \mathbb{G}$. Denote by $\mathbb{D}$ a convex bounded polygon in the plane and by a_1, a_2,..., a_n sides of $\mathbb{D}$. Then

$$[\mathbb{D}] = \{g \in \mathbb{G} : \; g \cap \mathbb{D} \neq \emptyset\} = \bigcup_{i<j} ([a_i] \cap [a_j])$$

where $[a_i] \cap [a_j]$ is the set of lines hitting both sides a_i and a_j of $\mathbb{D}$.

Let b_{ij} be the distance between the parallel segments a_i and a_j (i.e. the distance between the lines containing a_i and a_j). Further, $h(\varphi) \neq b_{ij}$ is the height of the maximal parallelogram with two sides equal to $\chi(\varphi) = g(\varphi) \cap \mathbb{D}$, $g(\varphi) \in [a_i] \cap [a_j]$ ($g(\varphi)$ is a line with φ-direction), and the other two sides lie on the parallel sides a_i and a_j,

$$h(\varphi_\chi) = h\left(\arccos \frac{b_{ij}}{|\chi(\varphi)|}\right) + h\left(2\,\pi - \arccos \frac{b_{ij}}{|\chi(\varphi)|}\right).$$

Hence , $h(\cdot) = 0$ if the parallelogram is empty.

For the value of φ such that $|\chi(\varphi)| = y$ we have $h(\varphi_y) = h(\varphi_\chi)$.

We obtain

$$F(y) = 1 - \frac{1}{\sum\limits_{i=1}^{n} |a_i|} \left[\sum\nolimits_{i<j}^{I} \frac{y}{\sin \gamma_{ij}} \int_{\Phi_{ij}(y)} \sin \varphi \, \sin(\gamma_{ij} - \varphi) \, d\varphi - \right.$$

$$\left. - \sum\nolimits_{i<j}^{II} I_{ij}(y) \, h(\varphi_y) \, \frac{\sqrt{y^2 - b_{ij}^2}}{b_{ij}} + \sum_{i=1}^{n} (|a_i| - y)^+ \right], \qquad (2.1)$$

where $\sum^I$ is over all nonparallel pairs of segments a_i, $a_j \subset \partial\mathbb{D}$ and $\sum^{II}$ is over all parallel pairs of segments a_i and $a_j \subset \partial\mathbb{D}$, while $|a_i|$ is the length of a_i, $i = 1, ..., n$.

In the case where $\partial\mathbb{D}$ contains no pairs of parallel sides, formula (2.1) coincides with the expression given by R. V. Ambartzumian in [2], page 158.

It follows from (2.1) that to find distribution function $F(y)$ we have to calculate integrals of the form

$$\frac{1}{\sin \gamma} \int_{\Phi(y)} \sin \varphi \, \sin(\gamma - \varphi) \, d\varphi$$

for any two nonparallel segments a and b ($a \leq b$) with the angle γ between a and b (or their continuations) and also calculate the second sum in (2.1) (for pairs of parallel sides). Here $\Phi(y)$ is

$$\Phi(y) = \{\varphi : \text{ a chord joining } a \text{ and } b \text{ exists with direction } \varphi \text{ and length } y\}.$$

For the proof of (2.1) see [22] and [8].

In the classical Pleijel identities integration is over the measure in the space **G** of lines which is invariant with respect to the all Euclidean motions (see [1], [2]). In the paper [12] generalized Pleijel identities for any locally-finite, bundleless measure in the space **G** have been proved. These identities are applied to find the so-called orientation-dependent chord length distribution functions for bounded convex domains:

$$b(\phi_0) \cdot [1 - F_{\phi_0}(y)] = \frac{1}{2} \int_{[\mathbb{D}]} \delta(|\chi(g)| - y) \, |\chi(g)| \, |\sin(\phi - \phi_0)| \, dg -$$

$$-\frac{1}{2} \int_{[\mathbb{D}]} \delta'(|\chi(g)| - y) \cdot |\chi(g)|^2 \, |\sin(\phi - \phi_0)| \, \cot \alpha_1 \, \cot \alpha_2 \, dg, \qquad (2.2)$$

where $b(\phi)$ is the breadth function (ϕ is a direction), $\delta(y)$ is the Dirac's δ–function concentrated at y, α_1 and α_2 are the angles between $\partial\mathbb{D}$ and the line g at the endpoints of $\chi(g) = g \cap \mathbb{D}$ which lie in one half-plane with respect to the inside of $\mathbb{D}$, while $F_\phi(y)$ is the orientation-dependent distribution function at ϕ

$$F_{\phi_0}(y) = \frac{1}{b(\phi_0)} \mathcal{L}_1\{p : \, |\chi(p, \phi_0)| < y\},$$

where $\mathcal{L}_1$ is one dimensional Lebesgue measure.

In the case where $\mathbb{D}$ is a polygon with no parallel sides, we have the following formula (see [12]):

$$b(\phi_0) \cdot [1 - F_{\phi_0}(y)] =$$

$$= \frac{y}{2} \sum_{i<j} \frac{1}{\sin\gamma_{ij}} \int_{\Phi_{ij}(y)} |\sin(\phi - \phi_0)| \, (\cos\phi\cos(\gamma_{ij} - \phi) + 2\sin\phi\sin(\gamma_{ij} - \phi)) \, d\phi +$$

$$+ \sum_{i=1}^{n} |\sin(\phi_i - \phi_0)| \left[(a_i - y)^+ - \frac{a_i \, I(a_i \geq y)}{2} \right],$$

where a_i is the side of the polygon $\mathbb{D}$, $i = 1, 2, ..., n.\gamma_{ij}$ is the angle between nonparallel sides a_i and a_j (or their continuations), α_1 is the angle between a_i and $\chi(g) = g \cap D$, and α_2 is the angle between a_j and $\chi(g)$ $(g \in [a_i] \cap [a_j])$. and

$$\Phi_{ij}(y) = \{\varphi : \text{ a chord joining } a_i \text{ and } a_j \text{ exists which has direction } \varphi \text{ and its length is } y\}.$$

3 KNOWN PARTICULAR CASES

I) A DISC.

For the case of the disc of radius R distribution function has the following form:

$$F_{\phi_0}(y) = \begin{cases} 0 & \text{if } \; y \leq 0, \\ 1 - \frac{\sqrt{4R^2 - y^2}}{2R} & \text{if } \; y \in (0, 2R). \\ 1 & \text{if } \; y > 2R \end{cases} \tag{3.1}$$

The corresponding density function is

$$f_{\phi_0}(y) = \begin{cases} 0 & \text{if } \; y \notin (0, 2R), \\ \frac{y}{2R\sqrt{4R^2 - y^2}} & \text{if } \; y \in (0, 2R) \end{cases} \tag{3.2}$$

that coincides with the form of "mixed" density function $f(y)$ (see [10]).

II) A REGULAR TRIANGLE.

In the case of a regular triangle of side a, density function has the form:

$$f(y) = \begin{cases} 0, & \text{if} \quad y \notin [0, a] \\ \frac{1}{2a} + \frac{\pi}{3\sqrt{3}a}, & \text{if} \quad 0 < y \leq \frac{\sqrt{3}a}{2} \\ -\frac{\sqrt{4y^2-3a^2}}{2\,y^2} + \frac{1}{2a} + \frac{2}{\sqrt{3}a} \arcsin\frac{a\sqrt{3}}{2y} - \frac{2\pi}{3\sqrt{3}a}, & \text{if} \quad \frac{\sqrt{3}a}{2} < y \leq a \end{cases} \tag{3.3}$$

and distribution function has the form

$$F(y) = \begin{cases} 0, & \text{if} \quad y \leq 0 \\ \left(\frac{1}{2} + \frac{\pi}{3\sqrt{3}}\right) \frac{y}{a}, & \text{if} \quad 0 \leq y \leq \frac{\sqrt{3}a}{2} \\ \frac{y}{2a} - \frac{2\pi}{3\sqrt{3}}\frac{y}{a} + \frac{2y}{\sqrt{3}a} \arcsin\frac{a\sqrt{3}}{2y} + \frac{\sqrt{4y^2-3a^2}}{2y}, & \text{if} \quad \frac{a\sqrt{3}}{2} \leq y \leq a \\ 1, & \text{if} \quad y \geq a. \end{cases} \tag{3.4}$$

The result coincides with the result by Sulanke ([6] see also [10]).

III) A RECTANGLE.

In the case of rectangle with sides a and b, $a \leq b$, density function of the length of a random chord has the form:

$$f(y) = \begin{cases} 0, & \text{if} \quad y \notin [0, \sqrt{a^2+b^2}) \\ \frac{1}{a+b}, & \text{if} \quad 0 < y \leq a \\ \frac{a^2\,b}{(a+b)y^2\,\sqrt{y^2-a^2}}, & \text{if} \quad a < y \leq b \\ \frac{ab}{(a+b)y^2}\left(\frac{a}{\sqrt{y^2-a^2}} + \frac{b}{\sqrt{y^2-b^2}}\right) - \frac{1}{a+b}, & \text{if} \quad b < y \leq \sqrt{a^2+b^2}, \end{cases} \tag{3.5}$$

while distribution function $F(y)$ for a rectangle has the form

$$F(y) = \begin{cases} 0, & \text{if} \quad y \leq 0 \\ \frac{y}{a+b}, & \text{if} \quad 0 \leq y \leq a \\ \frac{a}{a+b} + \frac{b}{a+b}\frac{\sqrt{y^2-a^2}}{y}, & \text{if} \quad a \leq y \leq b \\ 1 - \frac{y}{a+b} + \frac{1}{(a+b)y}\,(a\sqrt{y^2-b^2} + b\sqrt{y^2-a^2}), & \text{if} \quad b \leq y \leq \sqrt{a^2+b^2} \\ 1, & \text{if} \quad y \geq \sqrt{a^2+b^2}. \end{cases} \tag{3.6}$$

The result coincides with the result by W. Gille ([7] see also [10]).

IV) A REGULAR PENTAGON.

The chord length distribution function for a regular pentagon has the form:

$$F(y)=\begin{cases}0, & \text{if} \quad y\le 0\\ \frac{y}{a}\left(\frac{1}{2}-\frac{\pi(\sqrt{5}-1)}{5\sqrt{10+2\sqrt{5}}}\right) & \text{if} \quad 0\le y\le a\\ 1+\frac{\pi\,(5+3\,\sqrt{5})\,y}{5a\,\sqrt{10+2\sqrt{5}}}-\frac{2\,(\sqrt{5}+1)\,y}{a\,\sqrt{10+2\sqrt{5}}}\,\arcsin\frac{a\,\sqrt{10+2\sqrt{5}}}{4y}- & \\ -\frac{\sqrt{5}+1}{2}\,\cos\left(\arcsin\frac{a\,\sqrt{10+2\sqrt{5}}}{4y}\right), & \text{if} \quad a\le y\le \frac{a}{2}\,\sqrt{5+2\sqrt{5}}\\ 1+\frac{\pi\,(5+3\,\sqrt{5})\,y}{5a\,\sqrt{10+2\sqrt{5}}}-\frac{2\,(\sqrt{5}+1)\,y}{a\,\sqrt{10+2\sqrt{5}}}\,\arcsin\frac{a\,\sqrt{10+2\sqrt{5}}}{4y}- & \\ -\frac{\sqrt{5}+1}{2}\,\cos\left(\arcsin\frac{a\,\sqrt{10+2\sqrt{5}}}{4y}\right)- & \\ -\frac{(\sqrt{5}+1)^2 y}{a\sqrt{10+2\sqrt{5}}}\,\arccos\frac{a\,\sqrt{10+2\sqrt{5}}}{2y(\sqrt{5}-1)}+ & \\ +\frac{2(\sqrt{5}+1)}{(\sqrt{5}-1)^2}\,\sin\left(\arccos\frac{a\,\sqrt{10+2\sqrt{5}}}{2y\,(\sqrt{5}-1)}\right), & \text{if} \quad \frac{a}{2}\,\sqrt{5+2\sqrt{5}}\le y\le \frac{\sqrt{5}+1}{2}\,a\\ 1, & \text{if} \quad y\ge \frac{\sqrt{5}+1}{2}\,a,\end{cases} \tag{3.7}$$

where a is the side of a regular pentagon.

It is not difficult to verify that this function is continuous function.

The density function has the form:

$$f(y)=\begin{cases}0, & \text{if} \quad y\notin\left[0,\frac{\sqrt{5}+1}{2}\,a\right]\\ \frac{1}{a}\left(\frac{1}{2}-\frac{\pi(\sqrt{5}-1)}{5\sqrt{10+2\sqrt{5}}}\right), & \text{if} \quad 0<y\le a\\ \frac{\pi\,(5+3\,\sqrt{5})}{5a\,\sqrt{10+2\sqrt{5}}}-\frac{2\,(\sqrt{5}+1)}{a\,\sqrt{10+2\sqrt{5}}}\,\arcsin\frac{a\,\sqrt{10+2\sqrt{5}}}{4y}+ & \\ +\frac{2\,(\sqrt{5}+1)}{\sqrt{16y^2-a^2(10+2\sqrt{5})}}-\frac{a^2\,(5+3\,\sqrt{5})}{2\,y^2\,\sqrt{16\,y^2-a^2(10+2\sqrt{5})}}, & \text{if} \quad a<y\le \frac{a}{2}\,\sqrt{5+2\sqrt{5}}\\ \frac{\pi\,(5+3\,\sqrt{5})}{5a\,\sqrt{10+2\sqrt{5}}}-\frac{2\,(\sqrt{5}+1)}{a\,\sqrt{10+2\sqrt{5}}}\,\arcsin\frac{a\,\sqrt{10+2\sqrt{5}}}{4y}+ & \\ +\frac{2\,(\sqrt{5}+1)}{\sqrt{16y^2-a^2(10+2\sqrt{5}}}-\frac{a^2\,(5+3\,\sqrt{5})}{2\,y^2\,\sqrt{16\,y^2-a^2(10+2\sqrt{5})}}- & \\ -\frac{(\sqrt{5}+1)^2}{a\sqrt{10+2\sqrt{5}}}\,\arccos\frac{a\,\sqrt{10+2\sqrt{5}}}{2y(\sqrt{5}-1)}- & \\ -\frac{(\sqrt{5}+1)^2}{\sqrt{4\,y^2(\sqrt{5}-1)^2-a^2(10+2\sqrt{5})}}+ & \\ +\frac{4\,a^2(5+3\,\sqrt{5})}{y^2\,(\sqrt{5}-1)^3\,\sqrt{4\,y^2(\sqrt{5}-1)^2-a^2(10+2\sqrt{5})}}, & \text{if} \quad \frac{a}{2}\,\sqrt{5+2\sqrt{5}}<y\le \frac{\sqrt{5}+1}{2}\,a\end{cases} \tag{3.8}$$

The result coincides with the result by V.K. Ohanyan and N. G. Aharonyan ([8]).

V) A REGULAR HEXAGON.

In the paper [9] H. S. Harutyunyan calculated explicit expression for chord length distribution function for a regular hexagon:

$$F(y)=\begin{cases}0, & \text{if } \ y\le 0,\\ \frac{y}{a}\left(\frac{1}{2}-\frac{\pi}{6\sqrt{3}}\right), & \text{if } \ 0<y\le a,\\ 1+\frac{\pi\, y}{2a\sqrt{3}}-\frac{2\,y}{a\,\sqrt{3}}\ \arcsin\frac{a\,\sqrt{3}}{2\,y}-\frac{\sqrt{4y^2-3a^2}}{2y}, & \text{if } \ a<y\le a\,\sqrt{3},\\ 1+\frac{y}{a}\left(\frac{\pi}{6\sqrt{3}}-\frac{1}{2}\right)-\frac{y}{a\,\sqrt{3}}\ \arccos\frac{a\,\sqrt{3}}{y}+\frac{2\,\sqrt{y^2-3a^2}}{y}, & \text{if } \ a\,\sqrt{3}<y\le 2\,a,\\ 1, & \text{if } \ y>2\,a,\end{cases} \tag{3.9}$$

where a is the side of a regular hexagon.

The density function has the form:

$$f(y)=\begin{cases}0, & \text{if } \ y\notin(o,2\,\pi],\\ \frac{1}{a}\left(\frac{1}{2}-\frac{\pi}{6\sqrt{3}}\right), & \text{if } \ 0<y\le a,\\ \frac{\pi}{2a\sqrt{3}}-\frac{2}{a\,\sqrt{3}}\ \arcsin\frac{a\,\sqrt{3}}{2\,y}+\frac{\sqrt{4y^2-3a^2}}{2y^2}, & \text{if } \ a<y\le a\,\sqrt{3},\\ \frac{1}{a}\left(\frac{\pi}{6\sqrt{3}}-\frac{1}{2}\right)-\frac{1}{a\,\sqrt{3}}\ \arccos\frac{a\,\sqrt{3}}{y}+\frac{6\,a^2-y^2}{y^2\,\sqrt{y^2-3a^2}}, & \text{if } \ a\,\sqrt{3}<y\le 2\,a.\end{cases} \tag{3.10}$$

4 Conclusion

The chord length distribution function $F(y)$ are independent of the positions of the domains in the plane, thus it coincides for congruent domains.

Assume now to have the information about the distribution of the chord length not in the “completely mixed” form, but separated direction by direction. One can show that the problem of determining a body from this data, gave a positive answer using “orientation dependent” chord length distribution, when $\mathbb{D}$ is a planar convex polygon (see W. Nagel [13]) and for any bounded convex domain (see [20]). In fact, the covariogram problem is equivalent to the problem of determining a convex body from all its separate chord length distributions (see [13], [20]). A set of four directions whose slopes have a transcendental cross ratio will ensure that the corresponding parallel X-rays determine each planar convex domain (see [21]). The following problem arise.

OPEN PROBLEM. Does there exist a finite set of directions $V = \{\phi_1, ..., \phi_m\}$ such that the corresponding set of "orientation-dependent" chord length distribution functions $F_{\phi_1}(y)$, ..., $F_{\phi_n}(y)$ determine a bounded convex domain uniquely. Find minimal m which satisfies this condition.

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