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ON DIVISIBILITY TESTS AND THE CURRICULUM DILEMMA

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Abstract

Constructing rules for determining the divisibility of whole numbers is a subject that is being dropped from the curriculum. Few teachers today know if a simple division of whole numbers will give an integer quotient without doing the actual division, nor do they have the skills to construct such a test. This paper takes the stance that this is a shame, and it tries to rectify the situation by presenting several different methods for constructing rules of divisibility. Some of the methods presented are known but not well-known, while others are completely new; yet all are within the grasp of elementary school teachers.

Keywords: Divisibility rules, divisibility criteria, whole number division, proof.

1. Background: two sides to the argument. Calculators and computers have changed the school mathematics curriculum to the point where even experienced teachers are pondering the merits and demerits of drill and practice ($D\&P$), a notion that used to stand at the heart of the curriculum for "cementing in" ideas. Mathematics educators have been debating the role of $D\&P$ for years, with each side giving passionate arguments as to why their thinking on the subject should be adopted by curriculum decision makers, and by those who are still uncertain as to which side of the polemic to endorse. Do $D\&P$ really help students develop a deeper understanding and appreciation for the notions under study, or is it as the adversaries claim, "boring to the students," and a major contributing factor as to why students hate mathematics? The answer to this will never be definitive within our profession as a whole, but each teacher must answer this question for themself. Where does one draw the line these days with respect to $D\&P$, taking into account the existence of sophisticated computer algebra systems where most of the problems encountered in school mathematics can be solved in nanoseconds, if one knows how to set up the computer to solve them?

Every teacher has lists of essential topics and skills they believe all children should know, but unfortunately, the topics on these lists and the depth of the knowledge we wish to impart to the children concerning these topics are not standard, even with respect to the simplest of notions. E.g., should children be expected to know how to multiply a three digit whole number by a two digit whole number? I, and thousands of other teachers, say "yes"; but I am certain that just as many teachers can be found supporting a negative answer on this topic–and they hold to this negative stance even when the issue is phrased in a personal way: do you want your children (or grandchildren) to be able to correctly carry out the work long-hand to compute (538)(79)? To my dismay, I have colleagues who claim that they do not care whether or not their own children can carry out such a multiplication. They validate this stance by saying that we are living in the 21^{st} century, where calculators and computers are everywhere; they can be found even on one's wristwatch. And what about $\frac{1}{2}+\frac{1}{3}$? (They seem not to care about this either!)

2. A rationale for studying rules of divisibility. Probably the main reasons for studying rules of divisibility are to help students experience the thrill of creating mathematics, and to give one a sense of intellectual accomplishment. I doubt if anyone these days is ever going to calculate whether or not 7 divides 7568940320156 by hand to see if one obtains a whole number quotient. But developing the machinery to answer this question is another matter.

Listed below are a few different ways to know if $\frac{a}{b}=c$, where a, b, and c are whole numbers ($b \neq 0$). The methods presented in sections 3 and 4 are known but not well known; however I believe the rules presented in section 5 are completely new.

3. Rules of divisibility. Most students can tell if a natural number is divisible by $2, 3, 4, 5, 6, 8, 9$, and 10. For completeness, we list the rules below and make brief comments on some of them.

- **For 2:** A natural number $N = abcde$ is divisible by 2 if its last digit is even.
- **For 3:** A natural number $N = abcde$ is divisible by 3 if the sum of its digits is divisible by 3. E.g., $3\Big|(a+b+c+c+d+e)$. This is easy to see by writing the number N in expanded form:

$$\begin{aligned} N &= abcde \\ &= a(10^4)+b(10^3)+c(10^2)+d(10)+e \\ N &= a(9999+1)+b(999+1)+c(99+1)+d(9+1)+e \\ &= \underbrace{a\cdot 9999+b\cdot 999+c\cdot 99+d\cdot 9}+\underbrace{a+b+c+d+e} \end{aligned}$$

It is well known that for whole numbers N, m and n where $N = m+n$ and for prime number p, if p divides N, and p divides m, then p also divides n. I.e.,

$$\begin{aligned} N &= m+n \\ p\big|N \text{ and } p\big|m &\longrightarrow p\big|n. \end{aligned}$$

There are methods to rigorously prove this statement, but students readily accept it in an intuitive manner without proof. So in summary,

$$If\ 3\Big| N \text{ and } \mathit{3}\Big| \underbrace{a\cdot 9999 + b\cdot 999 + c\cdot 99 + d\cdot 9}, \text{ then } \mathit{3}\Big| \underbrace{a+b+c+d+e}.$$

I.e., if 3 divides N, then 3 divides the sum of the digits in N, and vice versa.

- **For 4:** A natural number $N = abcde$ is divisible by 4 if the number formed by the last two digits of N is divisible by 4. E.g., if $4\Big|de$, then $4\Big|N$.

  This is also easy to see by writing the number N in a slightly different form.

$$\begin{aligned} N &= abcde \\ &= a(10^4) + b(10^3) + c(10^2) + d(10) + e \\ &= \underbrace{a(10^4) + b(10^3) + c(10^2)} + \underbrace{d(10) + e} \end{aligned}$$

  The number 4 divides all terms on the right hand side of the above equality except the two digit number de. So if 4 divides the two digit number formed by de, then the number N is divisible by 4.

- **For 5:** A natural number $N = abcde$ is divisible by 5 if the last digit of N is either a 5 or a zero. This rule is immediately obvious to students and it is generally accepted without elaboration.

- **For 6:** Here $N = abcde$ is divisible by 6 if N is divisible by both 2 and 3. I.e.

$$\text{If } \mathit{2}\Big| N \text{and if } \mathit{3}\Big| N, \text{then } \mathit{6}\Big| N.$$

  Again, there are rigorous ways to prove this statement, but students immediately accept the rule as being obvious.

- **For 7:** The test for 7 is a bit tricky, so we'll look at shortly.

- **For 8:** Here $N = abcde$ is divisible by 8 if the number formed by the last three digits of N is divisible by 8. That is, if the number cde is divisible by 8. The proof is very similar to the one for divisible by 4.

$$\begin{aligned} N &= abcde \\ &= a(10^4) + b(10^3) + c(10^2) + d(10) + e \\ &= \underbrace{a(10^4) + b(10^3)} + \underbrace{c(10^2) + d(10) + e} \end{aligned}$$

  The number $10^4 = 10000$ is divisible by 8 and so then is the number $a(10^4)$. Similarly for the number $b(10^3)$. But the number 10^2 is not divisible by 8. So writing N as

$$N = abcde$$

$$= a(10^4) + b(10^3) + c(10^2) + d(10) + e$$

$$= \underbrace{a(10^4) + b(10^3)} + \underbrace{c(10^2) + d(10) + e},$$

we see that the number 8 divides all terms on the right hand side of the equality above except the three digit number formed by cde. So if 8 divides the three digit number formed by cde, then the number N is divisible by 8. It should also be noted that since $8 \Big| a(10^4) + b(10^3)$, the number 8 will also divide the larger numbers

$$x(10^5) + a(10^4) + b(10^3),$$

$$y(10^6) + x(10^5) + a(10^4) + b(10^3), \text{ etc.}$$

So no matter how large the whole number is (when written in base 10), it will be divisible by 8 if the number formed by the last three digits of it is divisible by 8.

- **For 9:** The rule for divisibility by 9 is the same as it is for divisibility by 3, and it's proof follows immediately from it. A number N is divisible by 9 if the number formed by the sum of its digits is divisible by 9.

Back to divisibility by 7.

Method 1: The procedure developed here follows from the above by writing the whole number N in expanded form. Let

$$\begin{aligned} N &= abcdefgh \\ &= a10^7 + b10^6 + c10^5 + d10^4 + e10^3 + f10^2 + g10 + h \\ &= a\Bigg(1428571(7) + 3\Bigg) + b\Bigg(142857(7) + 1\Bigg) + c\Bigg(14285(7) + 5\Bigg) + d\Bigg(1428(7) + 4\Bigg) \\ &\quad + e\Bigg(142(7) + 6\Bigg) + f\Bigg(14(7) + 2\Bigg) + g\Bigg(1(7) + 3\Bigg) + h(1) \end{aligned}$$

Rearranging gives

$$\begin{aligned} N &= abcdefgh \\ &= a10^7 + b10^6 + c10^5 + d10^4 + e10^3 + f10^2 + g10 + h \\ &= \underbrace{a\Bigg(1428571(7)\Bigg) + b\Bigg(142857(7)\Bigg) + c\Bigg(14285(7)\Bigg) + d\Bigg(1428(7)\Bigg) + e\Bigg(142(7)\Bigg) + f\Bigg(14(7)\Bigg) + g\Bigg(1(7)\Bigg)} \\ &\quad \underbrace{+ a\Bigg(3\Bigg) + b\Bigg(1\Bigg) + c\Bigg(5\Bigg) + d\Bigg(4\Bigg) + e\Bigg(6\Bigg) + f\Bigg(2\Bigg) + g\Bigg(3\Bigg) + h\Bigg(1\Bigg)} \end{aligned}$$

So, if $7 \Big| \mathrm{N}$, then 7 must divide

$$\underbrace{a\Bigg(3\Bigg) + b\Bigg(1\Bigg) + c\Bigg(5\Bigg) + d\Bigg(4\Bigg) + e\Bigg(6\Bigg) + f\Bigg(2\Bigg) + g\Bigg(3\Bigg) + h\Bigg(1\Bigg)}.$$

We now do a trick: If $7 \Big| 4c$, then $7 \Big| (-3)(4c)$. And if $7 \Big| (-3)(4c)$ then $7 \Big| (4)(-3c)$. But since 7 does not divide 4, then we know that 7 **must divide** $(-3c)$. Ie.,

$$7 \Big| 4c \text{ implies that } 7 \Big| (-3\text{c}).$$

This "trick" goes under the name of "modular" arithmetic. We say $a \equiv b(\text{mod p})$ implies "a" when divided by "p" leaves the same remainder as when "b" is divided by "p". This is equivalent to saying that "p" divides "(a-b)". So with respect to the above: $4 \equiv -3(\text{mod } 7)$.

Returning to the above, 7 must divide $\underbrace{a\Big(3\Big) + b\Big(1\Big) + c\Big(5\Big) + d\Big(4\Big) + e\Big(6\Big) + f\Big(2\Big) + g\Big(3\Big) + h\Big(1\Big)}$ is equivalent to saying that 7 must divide

$$\underbrace{a\Big(3\Big) + b\Big(1\Big) + c\Big(-2\Big) + d\Big(-3\Big) + e\Big(-1\Big) + f\Big(2\Big) + g\Big(3\Big) + h\Big(1\Big)}.$$

$$\underbrace{\Big\{a(3) + b(1)\Big\} - \Big\{\text{c}(2) + \text{d}(3) + \text{e}(1)\Big\} + \Big\{\text{f}(2) + \text{g}(3) + \text{h}(1)\Big\}}.$$

We have just developed a test for divisibility by 7. We take a number $N = abcdefgh$ and starting on the right hand side, we block off the digits into groups of three.

$$\begin{aligned} N &= abcdefgh \\ &= ab\{cde\}\{fgh\} \end{aligned}$$

Starting with the units digit and moving from right to left, we multiply the units digit by 1, the digit in the tens place by 3, and the digit in the hundreds place by 2. We continue on doing this from right to left; we multiply the digit in thousands place by 1, the digit in the ten-thousands place by 3 and in the one-hundred thousands place by 2 and we continue on moving from right to left. We then take the algebraic sum as indicated by the formula we have developed. If this algebraic sum is divisible by 7, then the number N is divisible by 7. And if this algebraic sum is not divisible by 7, then N is not divisible by 7.

E.g., does $7 \Big| 56798435492$? Does 7 divide

$$\{3(5) + 1(6)\} + \{2(7) + 3(9) + 1(8)\} - \{2(4) + 3(3) + 1((5)\} + \{2(4) + 3(9) + 1(2)\}?$$

If so, then $7 \Big| 56798435492$.

This method of finding "a set of magic multipliers" for the digits in the blocked off number will always work. But although it works beautifully for developing a test for divisibility by 11, in general, it isn't very efficient, particularly for large prime divisors. Luckily for us, there is a more efficient method for deciding divisibility by 7 and other low-valued primes.

4. An efficient procedure. Let's again look at divisibility by 7. Suppose we want to know if $7 \mid N = abcdef$. Chopping off the units digit of N gives us the number $M = abcde$. So multiplying M by 10 and the adding on the units digit returns us to N. Ie.,

$$\begin{aligned} N &= abcdef \\ M &= abcde \\ 10M &= abcde0 \\ 10M + f &= abcde0 + f = N. \end{aligned}$$

Now we do a trick:

$$\begin{aligned} N &= abcdef \\ N &= abcde0 + f \\ N &= abcde0 + (21f - 20f) \\ N &= 10\Big(abcde - 2f\Big) + 21f. \end{aligned}$$

Now if 7 divides N then 7 divides the right hand side of the last equation above. The number 7 most certainly divides $21f$, and so 7 must also divide $10\Big(abcde - 2f\Big)$. But 7 does not divide 10, so this means that 7 must divide $\Big(abcde - 2f\Big)$.

We have just developed a procedure for determining if $7 \mid$ N. We chop off the units digit of N, multiply it by 2 and form the new number $abcde - 2f$. If 7 divides the new number, then 7 divides N; and if 7 does not divide $abcde - 2f$, then 7 does not divide N.

If N is large, then repeat the procedure. E.g. Does $7 \mid 56789341$?

$$7 \mid 56789341 \; ?$$

$$7 \mid 5678932 \; ?$$

$$7 \mid 567889 \; ?$$

$$7 \mid 56770 \; ?$$

$$7 \mid 5677 \; ?$$

$$7 \mid 553 \; ?$$

$$7 \Big| 49 \text{ ? yes!}$$

Therefore, $7 \Big| 56789341$.

This method of chopping off the units digit and then "playing" with the number as above always works, particularly with low valued primes, and students seldom have trouble with discovering divisibility rules by themselves. Here is another example: Find a procedure to determine if N is divisibly by 23.

Assume $N = abcdef$ then

$$\begin{aligned} N &= abcdef \\ M &= abcde \\ 10M &= abcde0 \\ 10M + f &= abcde0 + f \\ N &= 10M + f = abcde0 + f \\ N &= \qquad\qquad abcde0 + \big(70f - 3(23)f\big) \\ N &= \qquad\qquad 10(abcde + 7f) - 3(23)f. \end{aligned}$$

So if $23 \Big| N$ then 23 must divide the right hand side of the last equation above. The number 23 most certainly divides $-3(23)f$ so 23 must divide $10(abcde+7f)$. But 23 does not divide 10, so then 23 must divide $abcde + 7f$, and we have just developed a procedure for testing if a number N is divisible by 23.

Similar tests can be developed for low valued primes with unit digits of $\{1, 3, 7, 9\}$. Odd as it might seem, tests of divisibility for 7 and other low-valued primes are not well-known and few mathematics teachers seem to be able to construct them, even though Eisenberg (2000), Hoch (2001) and I, Nahir (2003), have tried to convince teachers to explicitly include them in their lessons–not for the rule itself, but for the thrill of discovering it. In this spirit, I now present an even more efficient way than what is presented in this section. I discovered these rules by musing on the above, and although they can be presented in an abstract manner, I present them here in a simplified way.

5. An even more efficient procedure for two-digit divisors.

5.1 For divisors ending in 1. Suppose that the units digit of a two digit divisor is 1 and that the units digit for N is u. We write the divisor d as $(10a + 1)$ and number N as $(10[N/(10)] + u)$, where $[XX]$ is the greatest integer function. Then $(10a + 1) \Big| N$ or

$$(10a + 1) \Big| (10[N/10] + u)$$

if, and only if, there is a number k such that

$$[N/10] - ua = k(10\text{a} + 1).$$

Proof: Suppose

$$\begin{aligned}
[N/10] - ua &= k(10a+1) \\
[N/10] &= ua + k(10a+1) \\
10[N/10] &= 10ua + 10k(10a+1) \\
10[N/10] + u &= 10ua + u + 10k(10a+1) \\
10[N/10] + u &= u(10a+1) + 10k(10a+1), \\
10[N/10] + u &= (10a+1)(u+10k) \text{ but} \\
10[N/10] + u &= N, \text{and so} \\
N &= (u+10k)(10a+1)
\end{aligned}$$

That is, N is divisible by $(10a+1)$ and the quotient is $q = (u+10k)$. Moreover, since each step in the above proof is reversible, the statement is indeed an if, and only if, theorem. And additionally, we obtain the quotient without doing the actual division!

Example 1: Does $61 \Big| 2257$? Here

$$\begin{aligned}
N &= 2257,\ u = 7,\ a = 6 \\
2257 &\rightarrow 225 - (7)(6) \ = \ 183 \ = \ k \cdot 61 \\
k &= 3 \text{ and thus, } 61 \Big| 2257.
\end{aligned}$$

To obtain the actual quotient q we work backwards:

$$q = (u+10k) = 7 + 10 \cdot 3 = 37, \text{ and indeed } 2257 = 37 \cdot 61.$$

Example 2: Does $71 \Big| 34293$? Since the expression $N - ua$ is not easily recognized as a multiple of the divisor $(10a+1)$, we use the procedure recursively as we did in Section 4. The idea is to a build a chain until we find an appropriate integer k that is a multiple of the divisor, and then work backwards to obtain the actual quotient.

$$\begin{aligned}
N &= 34293,\ u_0 = 3,\ a = 7 \\
34293 &\rightarrow 3429 - (3)(7) \ = \ 3408 \ = \ k_0 \cdot 71 \\
3408 &\rightarrow 340 - 8(7) = 284 = k_1 \cdot 71.
\end{aligned}$$

Here it is easy to see that $k_1 = 4$. So we know that $71 \Big| 34293$. To find the actual quotient q we simply calculate backwards.

$$\begin{aligned} k_1 &= 4, u_1 = 8, u_0 = 3 \\ k_0 &= u_1 + 10(k_1) = 8 + 40 = 48 \\ q &= u_0 + 10k_0 = 3 + 10(48) = 483. \text{ And indeed} \\ 34293 &= \Big(3 + 10(48)\Big)(71) = (483)(71). \end{aligned}$$

5.2 For divisors ending in 3. Suppose the two digit divisor is of the form $(10a+3)$. Then

$$(10a+3)\Big|(10[N/10]+u)$$

if, and only if, there is a number k such that

$$[N/10] - u(7a+2) = k(10\mathrm{a}+3).$$

Proof: Suppose

$$\begin{aligned} [N/10] - u(7a+2) &= k(10a+3) \\ [N/10] &= u(7a+2) + k(10a+3) \\ 10[N/10] &= 70ua + 20u + 10k(10a+3) \\ 10[N/10] + u &= 70ua + 21u + 10k(10a+3) \\ 10[N/10] + u &= 7u(10a+3) + 10k(10a+3), \\ 10[N/10] + u &= (10a+3)(7u+10k) \text{ but} \\ 10[N/10] + u &= N, \text{and so} \\ N &= (7u+10k)(10a+3) \end{aligned}$$

That is, N is divisible by $(10a+3)$ and the quotient is $q = (7u+10k)$.

Example 1: Does $43\Big|2881$? Here $u_0 = 1, a = 4$.

$$2881 \to 288 - (1)(30) \quad = \quad 258 = k_0 \cdot 43.$$

It is easy to see that $k_0 = 6$, and so $q = 7u_0 + 10k_0 = 67$, and $2881 = (67)(43)$.

Example 2. Does $53 \Big| 2575641$? Here $u_0 = 1, a = 5$. Using the greatest integer notation $[n]$ we see that

$$\begin{aligned}
N &\rightarrow [N/10] - u(7a+2) = k(10a+3) \\
2575641 &\rightarrow 257564 - (1)(37) = 25764 - 37 = 257527 = k_0(53) \\
257527 &\rightarrow 25752 - (7)(37) = 25752 - 259 = 25493 = k_1(53) \\
25493 &\rightarrow 2549 - (3)(37) = 2549 - 111 = 2438 = k_2(53) \\
2438 &\rightarrow 243 - 8(37) = 243 - 296 = -53 = k_3(53).
\end{aligned}$$

So $k_3 = -1$ and we know that $53 \Big| 2575641$. We obtain the actual quotient by computing the values of k_3, k_2, k_1, k_0 with resect to the various unit digits $u_0 = 1, u_1 = 7, u_2 = 3$ and $u_3 = 8$. Specifically,

$$\begin{aligned}
k_3 &= -1 \\
k_2 &= 7u_3 + 10k_3 = 7(8) + 10(-1) = 46 \\
k_1 &= 7u_2 + 10k_2 = 7(3) + 10(46) = 481 \\
k_0 &= 7u_1 + 10k_1 = 7(7) + 10(481) = 4859 \\
q &= 7u_0 + 10k_0 = 7(1) + 10(4859) = 48597
\end{aligned}$$

And indeed, $2575641 = 48597(53)$.

5.3 For divisors ending in 7. Suppose the two digit divisor is of the form $(10a+7)$. Then

$$(10a+3) \Big| (10[N/10] + u)$$

if, and only if, there is a number k such that

$$[N/10] - u(3a+2) = k(10\mathrm{a}+7).$$

Proof: Suppose

$$\begin{aligned}
[N/10] &= u(3a+2) + k(10a+7) \\
[N/10] &= 30ua + 20u + 10k(10a+7) \\
10[N/10] + u &= 30ua + 21u + 10k(10a+7) \\
10[N/10] + u &= 3u(10a+7) + 10k(10a+7)
\end{aligned}$$

$$
\begin{aligned}
10[N/10] + u &= 3u(10a+7) + 10k(10a+7), \\
10[N/10] + u &= (10a+7)(3u+10k) \text{ but} \\
10[N/10] + u &= N, \text{and so} \\
N &= (3u+10k)(10a+7)
\end{aligned}
$$

That is, N is divisible by $(10a+7)$ and the quotient is $q = (3u+10k)$.

Example 1: Does $67 \mid 871$? Here the units digit $u_0 = 1$. So,

$$871 \to 87 - (1)(20) \quad = \quad 67 = k_0(67)$$

Thus k=1 and the quotient $q = (3u_0 + 10k_0) = 3(1) + 10(1) = 13$. And indeed $871 = (13)(67)$.

Example 2. Does $87 \mid 2980098$?

$$
\begin{aligned}
N &\to [N/10] - u(3a+2) = k(10a+7) \\
2980098 &\to 298009 - (8)(26) = 297801 = k_0(87)(26); u_0 = 8, k_0 =? \\
297801 &\to 29780 - (1)(26) = 29754 = k_1(87); u_1 = 1, k_1 =? \\
29754 &\to 2975 - (4)(26) = 2871 = k_2(87); u_2 = 4, k_2 =? \\
2871 &\to 287 - 1(26) = 261 = k_3(87); u_3 = 1, k_3 =? \\
261 &\to 26 - 1(26) = 0(87) = k_4 = 0; u_4 = 1, k_4 = 0.
\end{aligned}
$$

And to get the actual quotient we reverse the steps with the values of k_4, k_3, k_2, k_1, k_0 and with unit digits $u_0 = 8, u_1 = 1, u_2 = 4, u_3 = 1$, and $u_4 = 1$. Specifically,

$$
\begin{aligned}
N &\to [N/10] - u(3a+2) = k \\
k_3 &= 3u_4 + 0k_4 = 3(1) + 0 = 3 \\
k_2 &= 3u_3 + 10k_3 = 3(1) + 10(3) = 33 \\
k_1 &= 3u_2 + 10k_2 = 3(4) + 10(33) = 342 \\
k_0 &= 3u_1 + 10k_1 = 3(1) + 10(343) = 3423 \\
q &= 3u_0 + 10k_0 = 3(8) + 10(3423) = 34254
\end{aligned}
$$

And indeed, $2980098 = 34254(87)$.

5.4 For divisors ending in 9. Suppose the two digit divisor is of the form $(10a+9)$. Then

$$(10a+9)\Big|(10[N/10]+u)$$

if, and only if, there is a number k such that

$$[N/10] - u(9a+8) = k(10\mathrm{a}+9).$$

Proof: Suppose

$$\begin{aligned}
[N/10] &= u(9a+8)+k(10a+9)\\
[N/10] &= 90ua+80u+10k(10a+9)\\
10[N/10]+u &= 90ua+81u+10k(10a+9)\\
10[N/10]+u &= 9u(10a+9)+10k(10a+9)\\
10[N/10]+u &= 9u(10a+9)+10k(10a+9),\\
10[N/10]+u &= (10a+9)(9u+10k) \text{ but}\\
10[N/10]+u &= N, \text{and so}\\
N &= (9u+10k)(10a+9)
\end{aligned}$$

That is, N is divisible by $(10a+9)$ and the quotient is $q=(9u+10k)$.

Example 1: Does $29\Big|1073$? Here the units digit $u_0=3$.

$$1073 \quad\rightarrow\quad 107-(3)(26)=k\cdot 29;$$

So, $u_0=3$ and $k_0=1$ which forces the quotient to $(9u+10k)=9(3)+10(1)=37$; and indeed $1073=37(29)$.

Example 2: Does $59\Big|2863329$? Here the units digit $u_0=9$.

$$\begin{aligned}
N &\rightarrow [N/10]-u(9a+8)=k(10a+9)\\
2863329 &\rightarrow 286332-(9)(53)=2863325-467=285855=k_0(59)\\
285855 &\rightarrow 25585-(5)(53)=28320=k_1(59)\\
28320 &\rightarrow 2832-(0)(53)=2832=k_2(59)\\
28328 &\rightarrow 283-2(53)=177=k_3(59).
\end{aligned}$$

and we see that $k_3 = 3$. Therefore, $59 \big| 2863329$.

In getting the quotient we note that the unit digits are $u_3 = 2, u_2 = 0, u_1 = 5, u_0 = 9$. And now we proceed as before:

$$
\begin{aligned}
N &\rightarrow [N/10] - u(9a + 8) = k(10a + 9) \\
k_3 &= 3 \\
k_2 &= 9u_3 + 10k_3 = 9(2) + 10(3) = 48 \\
k_1 &= 9u_2 + 10k_2 = 9(0) + 10(48) = 480 \\
k_0 &= 9u_1 + 10k_1 = 9(5) + 10(480) = 4845 \\
q &= 9u_0 + 10k_0 = 9(9) + 10(48450) = 48531
\end{aligned}
$$

And indeed, $2863329 = 48531(59)$.

6. In summary. In this article we have presented several different ways to test whether or not for whole numbers d and N, $d \neq 0$, if $d \big| N$ gives a whole number quotient. Such a unit of study used to be part of the school curriculum, but it has all but disappeared from it, and this paper takes the stance that it shouldn't. Students can gain much from studying such a unit, because hidden in it are doors of opportunity for having students experience the thrill of having created mathematical procedures by themselves, for asking why one method is preferred over another, and how the methods they have developed can be generalized. Granted, there are many such exercises that can be included in the curriculum to accomplish these lofty goals, but this particular topic seems to be one of the nicest, and one which is readily available to most students. My goal for writing this paper will be realized if a few of the readers will try teaching this topic to their students. I think that the types of thinking involved with this unit are particularly nice–and that the benefits from studying such a unit will spill over into other parts of the curriculum.

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