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ACCURATE SOLUTION ESTIMATE AND ASYMPTOTIC BEHAVIOR OF NONLINEAR DISCRETE SYSTEM

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This article deals with the nonlinear discrete dynamical system

$$x(t+1) = A(t)x(t) + f(t, x(t)), \quad t \in N_0 = \{0, 1, 2, ...\}$$

We first derive accurate estimate for the norm of solution of this system. This gives us stability condition and bound for the region of attraction of the stationary solution. We also give sufficient conditions for the asymptotic stability of the null solution of the above system. Our approach is based on the concept of generalized subradius for the coefficient matrices. Numerical example showing asymptotic behavior of the null solution is also given to support our result.

Keywords: generalized subradius ; asymptotic stability; discrete Gronwall's inequality.

1. Introduction

Let R^n be the set of n-real vectors endowed with the Euclidean norm $\| \, . \, \|$. Consider the n dimensional discrete system

$$x(t+1) = A(t)x(t) + f(t, x(t)), \quad t \in N_0 = \{0, 1, 2, ...\} \tag{1}$$

where $(A(t))_{t \in N_0}$ is a sequence of real $n \times n$ matrices and $\{f(t, x(t)\}_{t \in N_0}$ is a sequence of nonlinear continuous vector functions.

Medina and Gil derived accurate estimates for the norms of solutions using the approach based on "freezing" method for difference equations and on recent estimates for the powers of a constant matrix (see [Medina and Gill (2004)]). Also many authors have studied the asymptotic stability of the null solution of such systems.

A well-known result of Perron which dates back to 1929 (see [Gordon (1972)], [Ortega (1973)] , page 270 and [LaSalle (1976)] , Theorem 9.14) states that system (1) is asymptotically stable with $A(t) \equiv A$ (constant matrix) provided that spectral radius $\rho(A)$ of A is less than 1 and $f(t,x) = o(\parallel x \parallel)$. We established asymptotic stability of the null solution using a concept of (sp) matrix and taking some growth condition on f (see [George, Shah (to appear)])

To obtain the estimates for the norms of solution and asymptotic stability of the null solution of system (1), we use the recent concept of generalized subradius.

Czornik introduced the the ideas of generalized spectral subradius and the joint spectral subradius and shown the relationship between generalized spectral radii and the stability of discrete time-varying linear system (refer [Czornik (2005)])

Let Σ denote a nonempty set of real $n \times n$ matrices. For $m \geq 1$, Σ^m is the set of all products of matrices in Σ of length m,

$$\Sigma^m = \{A_1 A_2 ... A_m : A_i \in \Sigma, \;\; i = 1, 2, ..., m\}$$

Denote by $\rho(A)$ the spectral radius and by $\parallel A \parallel$ a matrix norm of the matrix A. Let $A \in \Sigma^m$.

Definition 1.1. *The generalized spectral subradius of* Σ *is defined as*

$$\tilde{\rho_*}(\Sigma) = \inf_{m \geq 1} (\inf_{A \in \Sigma^m} \rho(A))^{\frac{1}{m}}.$$

Definition 1.2. *The joint spectral subradius of* Σ *is defined as*

$$\hat{\rho}_*(\Sigma) = \inf_{m \geq 1} (\inf_{A \in \Sigma^m} \parallel A \parallel)^{\frac{1}{m}}.$$

Czornik proved that the generalized spectral subradius and joint spectral subradius are equal for any nonempty set Σ and the common value of $\tilde{\rho_*}(\Sigma)$ and $\hat{\rho}_*(\Sigma)$ is called the generalized subradius of Σ and will be denoted by $\rho_*(\Sigma)$ (see [Czornik (2005)]).

Let

$$\Phi(t,0) = A(t-1)A(t-2)...A(0), \;\; \Phi(t,t) = I$$

denotes the fundamental matrix of the system

$$x(t+1) = A(t)x(t) \quad , t \in N_0 \tag{2}$$

and

$$x(t,0,x_0) = \Phi(t,0)x_0$$

is the unique solution of equation (2) with initial condition $x(0,0,x_0) = x_0$.
In [Czornik (2005)], the following result is proved.

Proposition 1.3. *([Czornik (2005)]) Consider the discrete time-varying linear system (2). where $(A(t))$ is a sequence of matrices taken from Σ. Then there exists a sequence $(A(t))$ such that for any initial state $x_0 \in R^n$, we have $\lim_{t\to\infty} x(t) = 0$ if and only if $\rho_*(\Sigma) < 1$.*

Note that if $\rho_*(\Sigma) < 1$, then for fixed $d \in (\rho_*(\Sigma), 1)$ there exist matrices $A_{i_1}, A_{i_2}, ..., A_{i_{n_0}} \in \Sigma$ such that $\| A_{i_1} A_{i_2} ... A_{i_{n_0}} \| < d$.

We recall that the zero solution of (1) is stable if for any $\epsilon > 0$ there exist $\delta > 0$ and $t_0 \in N_0$ such that

$$\| x(t) \| < \epsilon$$

whenever $t \geq t_0$ and $\| x_0 \| < \delta$. We say that the zero solution of (1) is absorbing if for any $x_0 \in R^n$,

$$\lim_{t\to\infty} \| x(t) \| = 0.$$

We say that the zero solution of (1) is asymptotically stable if it is both stable and absorbing.
We extend the above result for the nonlinear system (1) under the following assumptions.

Assumptions :

(a) Let $\rho_*(\Sigma) < 1$ and

$$\| \Phi(t,j) \| \leq \beta_j, \quad for \ \ all \ \ 0 \leq j \leq t < \infty.$$

where $\sum_{j=0}^{\infty} \beta_j \leq \eta$ for some constant $\eta > 0$.

(b) There exist constants $q, \mu \geq 0$ such that

$$\| f(t,x) \| \leq q \| x \| + \mu, \ \ \forall x \in B_r = \{x \in R^n : \| x \| < r\} \ \ for \ \ some \ \ r > 0.$$

(c) The constants q and η defined above satisfy $q\eta < 1$.

2. Main Results

Now we derive accurate estimate for the norm of solution of system (1).

Theorem 2.1. *Under Assumptions (a), (b) and (c) any solution $\{x(t)\}_{t=0}^{\infty}$ of (1) satisfies the inequality*

$$\sup_{t\geq 0} \| x(t) \| \leq \frac{\beta_0 \| x_0 \| + \mu\eta}{1 - q\eta}$$

provided that

$$\| x_0 \| \leq \frac{r(1-q\eta) - \mu\eta}{\beta_0} \ \ for \ \ some \ \ r > 0$$

Proof. By inductive arguments, it is easy to see that the unique solution $\{x(t)\}_{t=0}^{\infty}$ of (1) under initial condition $x(0) = x_0$ is given by

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j))$$

i.e.

$$\| x(t) \| \leq \| \Phi(t,0) \| \| x_0 \| + \sum_{j=0}^{t-1} \| \Phi(t,j+1) \| (q \| x(j) \| + \mu) \quad if \quad \| x \| < r$$

Hence,

$$\| x(t) \| \leq \beta_0 \| x_0 \| + q \max_{i=0,1,\ldots t-1} \| x(i) \| \sum_{j=0}^{t-1} \| \Phi(t,j+1) \| + \mu \sum_{j=0}^{t-1} \| \Phi(t,j+1) \|$$

i.e.

$$\| x(t) \| \leq \beta_0 \| x_0 \| + q \max_{i=0,1,\ldots t-1} \| x(i) \| \sum_{j=0}^{\infty} \| \Phi(t,j+1) \| + \mu \sum_{j=0}^{\infty} \| \Phi(t,j+1) \|$$

i.e.

$$\max_{i=0,1,\ldots t} \| x(i) \| \leq \beta_0 \| x_0 \| + q\eta \max_{i=0,1,\ldots t} \| x(i) \| + \mu\eta$$

$$\sup_{i \geq 0} \| x(i) \| (1 - q\eta) \leq \beta_0 \| x_0 \| + \mu\eta$$

i.e.

$$\sup_{i \geq 0} \| x(i) \| \leq \frac{\beta_0 \| x_0 \| + \mu\eta}{(1 - q\eta)}$$

provided that

$$\| x_0 \| \leq \frac{r(1 - q\eta) - \mu\eta}{\beta_0}$$

Hence the theorem. □

We now prove the asymptotic stability of the null solution of system (1).

Theorem 2.2. *Suppose that the system (1)satisfies*

(i) $lim_{t\to\infty} \sum_{j=0}^{t-1} \| \Phi(t,j+1) \| = 0$
(ii) $\rho_*(\Sigma) < 1$ *and*
(iii) Assumption (b).

Then the zero solution of (1) is asymptotically stable.

Proof. The unique solution $\{x(t)\}_{t=0}^{\infty}$ of (1) under initial condition $x(0) = x_0$ is given by

$$x(t) = \Phi(t,0)x_0 + \sum_{j=0}^{t-1} \Phi(t,j+1)f(j,x(j))$$

Hence

$$\begin{aligned}\| x(t) \| &\leq \| \Phi(t,0) \|\| x_0 \| + \sum_{j=0}^{t-1} \| \Phi(t,j+1) \|\| f(j,x(j)) \| \\ &\leq \| \Phi(t,0) \|\| x_0 \| + \sum_{j=0}^{t-1} \| \Phi(t,j+1) \| (q \| x(j) \| + \mu)\end{aligned}$$

$$i.e. \| x(t) \| \leq \{\| \Phi(t,0) \|\| x_0 \| + \mu \sum_{j=0}^{t-1} \| \Phi(t,j+1) \|\} + \sum_{j=0}^{t-1} q \| \Phi(t,j+1) \|\| x(j) \|$$

Now applying the discrete Gronwall's inequality,

$$\| x(t) \| \leq \{\| \Phi(t,0) \|\| x_0 \| + \sum_{j=0}^{t-1} \mu \| \Phi(t,j+1) \|\} \prod_{j=0}^{t-1} [1 + q \| \Phi(t,j+1) \|]$$

Since condition (ii) and Proposition 1.3 implies that

$$\| \Phi(t,0) \|\| x_0 \| \rightarrow 0, \quad as \quad t \rightarrow \infty$$

Hence by using condition (i), we prove

$$\| x(t) \| \rightarrow 0 \quad as \quad t \rightarrow \infty$$

Hence the proof. □

3. Numerical Example

Example 3.1. *Consider the non-autonomous system given by the following equation.*

$$x(t+1) = A(t)x(t) + f(t,x(t)) \tag{3}$$

where the sequence $A(t) = \begin{pmatrix} \frac{1}{3}t & 0 \\ 1 & .04t \end{pmatrix}$, *Let* $\Sigma = \{A(h) : h = 0.002, 0.004, .006, ...\}$ *and*
$f = \frac{1}{6} \begin{pmatrix} \frac{1}{2}x_2(t) + \epsilon x_2^2(t) \\ \frac{1}{2}x_1(t) \end{pmatrix}$. *Let* $\epsilon \in (0, \frac{1}{2})$. *For* $n = 10$, *we can verify that the*

generalized spectral radius $\rho_*(\Sigma) = 0.008 < 1$ *and* $\eta = 1$. *Also*

$$\begin{aligned}
\| f(t,x) \|^2 &= \frac{1}{36}\{\frac{1}{4}x_2^2 + \epsilon x_2^2 + \epsilon^2 x_2^4 + \frac{1}{4}x_1^2\} \\
&\leq \frac{1}{36}\{\frac{1}{4}x_2^2 + \epsilon x_2^2 + \epsilon^2 x_2^2 + \frac{1}{4}x_1^2\} \\
&\leq \frac{1}{36}\{(\frac{1}{2} + \epsilon)^2 x_2^2 + \frac{1}{4}x_1^2\} \\
&\leq \frac{1}{36}(\frac{1}{2} + \epsilon)^2 \| x \|^2 \\
&\leq \frac{1}{36}d^2 \| x \|^2 \ \textit{taking} \ \ d = (\frac{1}{2} + \epsilon) < 1 \\
i.e., \| f(t,x) \| &\leq \frac{d}{6} \| x \| \\
\| f(t,x) \| &\leq q \| x \| \quad \textit{taking} \ \ q = \frac{d}{6}
\end{aligned}$$

i.e. f *satisfies the Assumption* (b) *with* $q = \frac{d}{6} = \frac{0.7}{6} = 0.1167 < 1$ *taking* $\epsilon = 0.2$ *without loss of generality and* $\mu = 0$. *The function* f *is continuous and it satisfies* $f(0) = 0$. *Note that* $q\eta = 0.1167 < 1$. *Hence all the Assumptions of Theorem (2.2) are satisfied. We can see in Figure* 1 *that initial state* $x_0 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$ *reaches to null state in* 10 *iterations. Hence the null solution of equation (3) is asymptotically stable.*

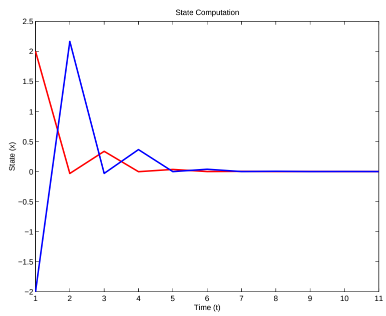

Fig. 1.

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