Sutra: International Journal of Mathematical Science Education

Vol. 1, No. 1, 1 – 8, 2008

MATHEMATICS THRILLERIUM

L. Radhakrishna
UGC Centre for Advanced Studies in Fluid Mechanics
Dept of Mathematics, Bangalore University, Bangalore 560001, India
lankaradhakrishna@yahoo.co.in

The following nine thrills are discussed: The chaos in mathematics communication, World War III and mathematics, definition of science/art from the 21st century definition of mathematics, the paralysis in analysis, 'EinSTein' and self orthogonal vectors, the Celerity monoid, 'EINstein' as a correction to 'NEWton' and Glashow's snake, 'EINSTein' and the Taj Mahal of Science, the mathematician's mathematician and the Strange Indian Melody.

1 INTRODUCTION

From 00 to 97: The prestige of Mathematics Education has been raised to the prestige of Mathematics Research only in the 21st century— on the eve of World Mathematics Year 2000. To be specific, the esteemed journal *Mathematical Reviews* published by the American Mathematical Society has given the Primary Subject Classification Number 97 to the discipline Mathematical Education in their revised Subject Classification Scheme of 2000; prior to 2000, education had no independent status as a primary branch of enquiry and was classified under the Secondary Subject Classification number 00A35 (00 for General). From 00 to 97 is a thrilling jump in status for research in mathematics education. Recently a computer journal has put it thus: *Education is business. Customer (student) is the king*.

The *SUTRA* Initiative*:* A perfunctory professor— who does his teaching work as a means of earning his livelihood— confines himself to the prescribed syllabus. He has no aptitude for creating an excited feeling for the subject in his audience. It needs effort to pep up interest in the class room with

- History of the concepts
- Philosophy of the reasoning in proofs.

However a professional professor takes care of minute details of presentation besides history and philosophy of the subject, so as to ensure a wonder-joy session; that is, start the lecture with wonder—a captivating caption— and end the lecture with joy due to the thrill of understanding. This is desirable but difficult and so perfunctory professors outnumber the professional ones. It is a tribute to the sagacity of the founders of the new journal *SUTRA*, for focusing on the educational aspects. The aim of this article, is to help the perfunctory teacher to evolve into a professional one via thrillerium—a typical collection of philosophical/historical thrills about mathematics. For instance one can replace the matter-of-fact title "Galois Group of a Field" in a class room, by the arresting caption "The Climax in Elegance in Algebra". It raises the attention curve of the students. We present in this small article some topics to add spice to pedagogy.

2 THE FIRST THRILL: The Chaos in Mathematics Communication

The Birth and Growth of Mathematics: Who invented the word 'mathematics'? Since 'history of mathematics' is not included in the syllabus, it is not a popular question in educational institutions, the said question becomes a quiz. The answer will help in appreciating how ancient the subject is. In fact it is the oldest discipline, since the word 'mathematics'—that which can be learnt— was invented by Pythagoras (584-495 B.C.). He coined the word

'philosophy' as well. Mathematics is 25 centuries old, while physical sciences are about one hundred years old. Biology, Statistics and Psychology are hardly 70 years old. The following table gives an estimate of the growth of the discipline in question:

Year	Number of Primary Subjects	Number of Secondary Subjects
500B.C.(Pythagoras era)	2	4
1868 A.D.	12	38
1979 A.D.	60	3400
2000 A.D.	67	14000

In the year 1980 about 200,000 theorems were published in 1600 journals [Davis and Hersh 1981]. The doubling period for mathematics being 10 years, in 2008 we safely estimate 800,000 theorems per annum in 14000 branches of mathematics [see the table above]. To verify the veracity of one theorem, a professional can take one day or even one month. This is an exponential proliferation.

The Chaos of Communication: It is not possible for one person to keep track of all the research in the 14000 specializations. With increasing technical jargon, the chaos in communication even among mathematicians is sufficiently complex! Consequently, how one can meaningfully convince the Caesars (the practitioners of the royal science of politics, who fund mathematics) — with the relevance and thrills of the subject— is much more challenging. A journal of education like *SUTRA* can provide partial order in the total chaos. Senior professionals may have to consider it a professional obligation to enlighten the funding agencies on the effectiveness of mathematics.

World War III and Mathematics: Science and technology provide an inexhaustible demand for more mathematical models. Linguistics and cliometry from the arts accelerate the thrill in the mathematization of culture. *Mathematics by fiat* is the clarion call from social welfare. From January 2000 Mathematical Reviews has introduced at Serial No.92 the major subject of research "Biology and other Natural Sciences". The successful mapping of the Human Genome Project in 2000A.D. is a tribute to mathematical techniques and theoretical computer science. Michael Atiyah eulogizes this ubiquitous role of mathematics thus: *The 20th century has transformed mathematics from a cottage industry run by a few semi-amateurs into a world-wide industry run by an army of professionals*" [P. viii, Arnold 2000] This army of professional mathematicians will play a major role in World War III, by monitoring the speed and accuracy of the delivery of weapon systems through efficient algorithms. Mathematics is the essence of algorithms— the soul of computers. It may be noted that World War I is credited to chemistry through trinitrotoluene (the gun powder) and World War II is credited to physics via release of atomic energy. Mathematics is more basic and ancient than physics/ chemistry and no wonder if it takes over World War III [Davis and Hersh 1981]. The education of such a prime discipline must be nurtured with priority. This is the purpose of SUTRA.

3 THE SECOND THRILL: The 21st Century Definition of Mathematics/ Science/Arts

"I love this definition…" declared the Fields Medalist David Mumford, in his address *Mathematics towards the Third Millennium* [P.199, Arnold 2000]. Since the particular description is popularized in the year 2000, we christen it as the 21st century definition, originally due to Davis and Hersh [P. 399, 1981]. It states that:

The study of mental objects with reproducible properties is called mathematics.

The thrills that follow from this definition are (i) if we replace the word *mental* by the word *physical*, we get the description of science. (ii) Keeping *mental* unchanged, if we replace *reproducible* by the word *irreproducible*, then we get the definition of arts.

Illustration: The elements $x, y, z, x * y, y * z$ of a set S, with the binary operation $*: S \times S \to S$, are mental objects, since x, y, z are all meaningless symbols. In calculus the functions $f, f', f'', \ldots$ with dashes representing derivatives are devoid of any meaning and so the symbols are mental objects. Since paintings and music are peculiar to its creator, the arts are not reproducible and so are subjective. Thus mathematics shares objectivity with science and it shares the study of nonphysical objects with the arts. However it is a paradox that mathematics *per se* is excluded from the Nobel Prize award unlike the sciences and the arts.

Note: Current dictionaries give definitions that are outdated by 4 centuries! For instance the Oxford dictionary (2003) cites the 16th century definition of mathematics:

The science of number, quantity, and space is called mathematics.

4 THE THIRD THRILL: THE PARALYSIS IN ANALYSIS

There are two perspectives on mathematics:

- The main stream approach
- The foundational approach.

In Section 3, we have followed the first perspective resulting in the latest definition of mathematics. The corresponding philosophy is named by Hersh as *Mathematical Humanism* [Grinstein and Lipsey 2001]. In this section we offer an exposition of the second approach. Foundations of mathematics deal with the laws of reasoning:

[R1] The Law of Excluded Middle: Every statement is either true or false.

[R2] The Law of Trichotomy: Every real number is either positive or negative or zero.

[R3] The Modified Law of Excluded Middle: Every statement is either true or false or

neither true nor false.

For describing the philosophies, we need the thrilling result:

[GP] Gödel's Proposition: Mathematics is fallible.

Here we recall the famous Incompleteness theorem due to Gödel which states that 'there is no algorithm whose output contains all true statements of arithmetic and no false ones' [P.311 Hersh 1997]. There are three beliefs on the subject:

[M1] Mathematics is invented.

[M2} Mathematics is discovered.

[M3] Mathematics that can be constructed in a finite number of steps is genuine.

Now we can describe the three warring groups owing allegiance to certain standard philosophies:

[P1] *Formalism* characterized by [R1], [R2] and [M1] with Hilbert as the exponent.

[P2] *Mathematical Platonism* featured by [R1], [R2], and [M2] with Einstein in the front

[P3] *Constructivism or Intuitionism* satisfying [R3], [GP] and [M3] with Brouwer as leader.

They respectively characterize pure mathematics, applied mathematics and computer science.

While Russell has demonstrated the falsity of [R1] through the construction of R-set, Brouwer has demonstrated the falsity of [R2] by constructing 'pi hat' [Davis and Hersh 1981]. It is well known that Cantor's theorem on the uncountability of real numbers is established by *reductio ad absurdum*, which is an indirect proof. Such proofs are based on the two valued logic of [R1], which is false for infinite sets. Besides the construction of real number system is not possible in a finite number of steps. This violates [M3]. Hence real analysis is not genuine mathematics according to [P3]. It is unfair mathematics as it uses the reasoning [R1], which is not true. Till a direct proof of the uncountability of real numbers is discovered by some genius in future, real analysis remains paralyzed. It follows that complex analysis and functional analysis share the paralysis at present. This is responsible for the birth of constructive functional analysis, circumventing the use of real numbers.

5 THE FOURTH THRILL: ''EinSTein" and the Self Orthogonal Vectors

The icon of the millennium: Washington based weekly *TIME* conducted an 'on line' poll from January 1999 to December 1999, to tell the world the Person of the 20th century and announced that Albert Einstein lead all the rest since he 'revised the universe with an equation'. The magazine identified the persons of each of the centuries from 11th to 20th (For instance Queen Elizabeth for 16th century, Sir Isaac Newton for 17th century) and compared their contributions to art, science, society etc. Again Einstein's influence was the most. Thus Einstein (1879-1955) became *the icon of the second millennium* (1001-2000) [Issacson *2000*]. We attempt a scintillating portrait of the 'most creative intellect of the human race' Einstein, by splitting the word 'Einstein' in three different ways and associate an appropriate mathematical thrill with each split.

"Ein ST ein": Einstein was a German and in German language 'ein' means 'one'. As the word Einstein starts with 'ein' and ends with 'ein', we infer that the name 'Ein ST ein' suggests that he obtained the unification or one-oneness of S and T. What do S and T in the middle could stand for? From history of mechanics, the obvious inference is that S refers to 'space' and T refers to 'time'! This unification of space-time was achieved in 1905 by Einstein. This amalgamation of S and T into ST is enshrined in his name 'EinSTein' itself. What a divine dispensation! From 1905 onwards 'events' form the basic data of physics and not just 'points'. Einstein's 'Interval'

$$\tau = \sqrt{(-x^2 - y^2 - z^2 + c^2t^2)} \tag{5.1}$$

between the events (0, 0, 0, 0) and (x, y, z, ct) where c is the velocity of light, became the focus of investigation rather than Euclid's 'Distance'

$$s = \sqrt{(x^2 + y^2 + z^2)} \tag{5.2}$$

between the points (0, 0, 0) and (x, y, z).

Self-orthogonal 4-vectors: From (5.2), we have

$$s^2 = 0 \text{ if and only if } x = 0, y = 0, z = 0, \tag{5.3}$$

but from (5.1) we deduce

$$\tau^2 = 0 \text{ if and only if } x^2 + y^2 + z^2 = c^2t^2. \tag{5.4}$$

For example the 4-vector $v = (x, 0, 0, ict)$ with $i^2 = -1$, is orthogonal to itself, since

$$v.v = (x^2 + 0^2 + 0^2 + i^2c^2t^2)$$

$$= 0, \text{ by (5.4).}$$

The inner product of v with itself vanishes and so it is self orthogonal. The concept of self orthogonal vectors is against common sense and so it is thrilling. The tangent to the light ray is self orthogonal.

6 THE FIFTH THRILL: EINSTEIN'S CELERITY MONOID

Let $(R, +, \cdot)$ denote the field of real numbers with usual addition, multiplication and c be a fixed privileged real number. The letter c is the first letter of the word '*celerity*' (speed). We define Einstein's composition (of high speeds)

$\oplus: R \times R \to R$

by $$(x, y) \to x \oplus y = (x + y) + (1 + (x \cdot y)c^{-2}). \qquad (5.5)$$

The composition $\oplus$ is a binary operation, since $x \oplus y$ is in R. We observe that

$$(x \oplus y) \oplus z = x \oplus (y \oplus z) \quad \forall x, y, z \in R \quad \text{(Associativity property)}$$

and $$x \oplus 0 = x = 0 \oplus x \qquad \forall x \in \quad \text{(Existence of identity).}$$

Hence we infer that $(R, \oplus)$ is a monoid.

Note the thrilling properties involving *c*:

(i) $c \oplus (-c) = 0 + 0$, which is indeterminate. (5.6)

(ii) The equation $c \oplus x = 0$ has no solution in x since it implies the untenable result $c = 0$; that is, the inverse of the privileged element *c* does not exist. So the monoid does not qualify to be a group.

(iii) $c \oplus x = c \qquad \forall x \in R.$

(iv) $c \oplus c = c.$ (5.7)

Bisociation: We recall a relation in real analysis analogous to (5.7). If *c* is the cardinality of the continuum, then [Simmons 1963]

$$c \blacksquare c = c \qquad (5.8)$$

where $\blacksquare$ represents the cardinal addition. Eq. (5.8) is often expressed as $\aleph_1 + \aleph_1 = \aleph_1$, with $\aleph_1$ as the uncountable actual infinity of real numbers. Such unexpected formal similarity in Equations (5.7) and (5.8) in different disciplines — high speed mechanics and real analysis—are referred to as 'bisociation', which is expected to be the dominant feature of 21st century mathematical research [Browder 2002]. In the algebra of electrical circuits and in Boolean algebra we have a similar result

$$x + x = x, \quad \forall x.$$

Remark 1: The equation (5.5) is the familiar relation in Einstein's special relativity for the composition of collinear velocities comparable to the velocity of light.

Remark 2: In Einstein's result (5.7), *c* is finite but huge equal to 1, 86,000 miles per hour, while in Cantor's result (5.8), c is the uncountable infinity. What a contrast in the scope of *c*!

Remark 3: With the potential infinity ∞ and the real number x the properties (iii) and (iv) are analogous to

$$\infty + x = \infty \text{ and } \infty + \infty = \infty.$$

7 THE SIXTH THRILL: 'EINstein' IS A CORRECTION TO 'NEWton'

In German 'ein' means 'one' and 'stein' means a 'stone' and a stone as a measure stands for 14 pounds. In England the new ton by weight stands for 2240 pounds. Obviously 14 is a correction (zero error) to 2240. This is the literal picture when we say: 'EIN stein' is a correction to 'NEW ton'. This thrilling interpretation of the two legendary names can be justified mathematically in more than one way. [Metaphorically also the contribution of Newton is more fundamental than Einstein. Without Newton's differential calculus of the 17th century, Einstein's tensor calculus of the 20th century would not have been possible.] Specifically to enlighten the small correction to Newton's formulae, we give two illustrations.

[1] Newton's famous formula for the force $\boldsymbol{F}$

$$\boldsymbol{F} = m\boldsymbol{a},$$

gets Einstein's modification for high speeds

$$\boldsymbol{F} = m\boldsymbol{a} + (\boldsymbol{F} \cdot \boldsymbol{u})\boldsymbol{u}c^{-2}.$$

The second term on the right hand side of this equation is negligible when u is small compared to c.

[2] In Newtonian mechanics, the planet Mercury describes an elliptical orbit given by the differential equation

$$D^2 u + u = mh^{-2}$$

where u is the reciprocal of the radial distance of the planet from the sun and D is the time derivative, m is the mass and h is a certain constant. Careful experiments have shown that Mercury's orbit was not a perfect ellipse. Einstein gave the corrected equation:

$$D^2 \text{ u} + \text{u} = \text{m}h^{-2} + 3mu^2c^{-2}.$$

The minuteness of the correction can be gauged, when it (the perihelion motion of Mercury) is found to be ***17 seconds of arc per century.*** One has to wait for a hundred years to see a slight deviation from elliptical orbit for the fastest planet Mercury.

8 THRILL SEVEN: 'EINstein' and GLASHOW'S SNAKE

The normal weight of the head of a person is 14 pounds or one stone and so the word 'EINstein' for 'one stone' can mean 'the one head' or 'the unique brain'. The head of Einstein has become photographers' delight, with his tousled mane and furrowed forehead. Encyclopedia Britannica (1960 edition) described Einstein as 'the most creative intellect of the human race'. Who will be remembered in the year 3001 A.D.? Certainly Einstein, since atom bomb, the Big Bang picture of the universe, and electronics, television, space travel bear his stamp.

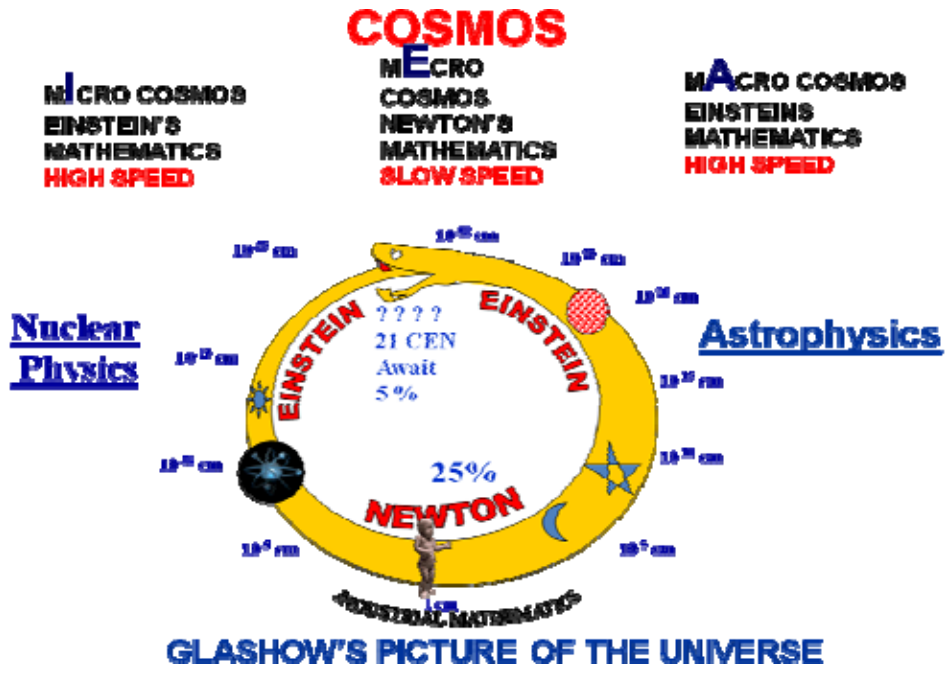
Einstein coined the words tensor, photon, and cosmos. Micro cosmos relevant to nuclear physics involves high speeds and so Einstein's mathematics of Special Relativity is relevant. Macro cosmos deals with super massive bodies at high speeds and at high pressures and the appropriate mathematical tool is Einstein's General Relativity. In the figure below, conceived by the Nobel laureate Glashow, micro cosmos occupies 35% of the universe and macro cosmos occupies 35%. It means 70% of the universe can be studied with Einstein's mathematics. For such a prolific individual, God must have designed the name in a special way!

In contrast, the slow speed mechanics of Newton is applicable to 25% of the cosmos, sometimes referred to as mecrocosmos— the universe accessible to the naked eye. This implies that mathematics of only 5% of the cosmos (the common portion between the hood and the tail of the snake *ouroborus*) remains to be discovered. In fact Einstein's theory fails at

Planck length 10^{-33}centimeters and at Planck time 10^{-44} seconds.

What happens below the Planck length and Planck time is a challenge for the 21st century scientists.

String theory is struggling to come to the rescue. We have to wait for an Einstein-like genius! They are so rare!



9 EIGHTH THRILL: 'EINSTein' AND TAJ MAHAL OF SCIENCE

The combination of the German words 'Einst' and 'ein' can be interpreted as 'the first one'. This refers to 'the first' discipline created entirely by just 'one man'. In other words the first person to create a subject without history in science is Einstein. The subject is General Relativity. In fact his happiest thought was [Narlikar 1978]

Gravitation ≡ Curvature of space-time.

He then announced the field equations of gravitating matter referred to as Taj Mahal of Science

(vide *Gravity* by George Gamow):

$$R_\beta^\alpha - \frac{1}{2} R \delta_\beta^\alpha = -8\pi G c^{-4} T_\beta^\alpha .$$

The left hand side is a divergence free tensor (known as the Einstein tensor) representing the geometry which is as beautiful and strong as the marble of the Taj Mahal. The right hand side is the stress-energy-momentum tensor of gravitating matter and π, G, c are usual constants. Consequently Einstein predicted in 1916 the then thrills:

(i) Light bends in the neighborhood of ponderable masses (verified by experiment in 1921)

(ii) Universe is expanding (verified by Hubble in 1919)

(iii) Planets have quirks in their elliptic orbits (not understood before 1916).

The experimental verification of Einstein's mathematical predictions on the existence of gravitational waves and Black holes is inconclusive to this day.

10 NINTH THRILL: THE MATHEMATICIAN'S MATHEMATICIAN RAMANUJAN

A scientist becomes immortal in the history of mathematics just by announcing one conjecture. Ramanujan (1887-1920) has written about 600 conjectures, some of which are still being checked for validity. This is a record and such natural mathematical talents are almost rare. Newman has called him 'The Mathematician's Mathematician' as distinguished from 'Poincare: The Physical Mathematician' [Newman 1972]. An exemplar of his genius eulogized as 'Strange Indian Melody' (an infinite continued fraction) by Hofstadter [1983] is reproduced below:

$$\frac{1}{1+}\,\frac{\exp(-2\pi\sqrt{5})}{1+}\,\frac{\exp(-4\pi\sqrt{5})}{1+}\cdots \text{to } \infty = \left(\sqrt{5}\,[1+A]^{-1} - B\right)\exp(2\pi/\sqrt{5})$$

where

$$A = \sqrt[5]{5^{3/4}\,\Omega - 1}\,,\quad B = \frac{(\sqrt{5}+1)}{2}\,,\quad ? = \left(\sqrt{5-1}/2\right)^{5/2}.$$

"He was a mathematician so great that his name transcends jealousies, the one superlatively great mathematician whom India produced in the last thousand years" [Kanigel 2003].

CONCLUSION

In this paper a sample of nine thrills have been designed out of author's teaching and research experience. The historical anecdotes add value to mathematical pedagogy. The philosophical discourses provide a unifying command of exposition. Every teacher must have experienced thrills while reading or explaining to students. These exciting facts could be shared for the benefit of future practitioners through the columns of *SUTRA*.

REFERENCES

Arnold et al (Ed) *Mathematics: Frontiers and Prospects,* American Mathematical Society, 2000.

Browder, F., Reflections on the Future of Mathematics, *Notices of the American Mathematical Society*, **49,** 658-663, 2002.

Davis, P.J. and Hersh R., *The Mathematical Experience*, Cambridge Birkhauser 1981.

Hersh, R., *What is Mathematics Really?* Oxford University Press, 1997.

Hersh, R 'Philosophical Perspectives in Mathematics Education' *in* Grinstein, L.S., and LipseyS.I. (Eds.) *Encyclopedia of Mathematics Education,* Routledge Falmer 2001.

Hofstadter, D.R., *Gödel, Escher, Bach: An Eternal Golden Braid,* Penguin Books, P.563, 1983.

Issacson, W. 'Person of the Century' in Knaeur (Ed) *Time 100,* Time Books, Volume III, 82-89, 2000.

Kanigel R., *The Man Who Knew Infinity: A Life of the Genius Ramanujan,* Abacus 2003.

Narlikar J. V., *Lectures on General Relativity and Cosmology*, Mac Millan (India), 1978.

Newman, E. 'Srinivasa Ramanujan' in Klein M. (Ed) *Mathematics in the Modern World: Readings from Scientific American,* Freeman, 1972.

Simmons G.F. Introduction to Topology and Modern Analysis, Mc Graw Hill, 1963.