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Ontology, Rough Y-Systems and Dependence

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In this research paper, we explore the philosophical connections between Rough Y-Systems RYS, mereology and concepts in applied ontology, introduce the concept of contamination-free rough dependence and compare this to possible concepts of probabilistic dependence. The nature of granular rough dependence is also characterized and the reason for breakdown of comparison of rough set models with probabilistic models are made clearer. From this we can test the validity of related comparisons in a semantic way.

Keywords: Rough Y-Systems; Applied Ontology; Rough Dependence; Rough Objects; Proto-Transitivity; Granulation; Axiomatic Theory of Granules; Contamination Problem; Probability Theory; Mereology.

1. Introduction

Vague concepts and contexts involving such concepts can be formally represented through rough sets. The concept of rough Y-systems (RYS) developed in [A.Mani (2011)], [A.Mani (2012a)] for a wide variety of reasons also provides a uniform framework for most if not all types of general rough sets. The approach is essentially an axiomatic approach to granulation that differs substantially from the *precision-based granules* approach.

Applied ontology is a way of applying the subject of ontology in philosophy to real-life situations towards improvement in representation, formalization and semantics. Some references are [Winston *et al.* (1987); Vieu (2007); Vieu and Aurnague (2007); Varzi (2006); Guarino (1998); Pratt and Lemon (1997); Keet (2008)]. The connections with applied ontology were not explicitly developed in earlier papers on RYS or even for other types of rough sets. So we develop the same in this paper towards improvement in formalization in application scenarios.

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Correspondences between semantics of rough systems have been studied in [A.Mani (2012b); A.Mani (2013a)] by the present author. In algebraic approaches, the motivations may be for the associated algebraic logic or duality or for direct understanding of semantics by way of comparison with other kinds of semantics. Some examples of the former class relate to morphisms between Nelson algebras in the literature [Pagliani and Chakraborty (2007); Sendlewski (1984)]. The latter class were introduced for RYS and special cases have been investigated by the present author in [A.Mani (2012b); A.Mani (2013a)]. In [A.Mani (2013b)], it was mentioned that correspondences between an approximate semantics of PRAX and a higher order semantics of PRAX could possibly be avoided by using dependence like predicates or functions. This is also a major motivation for considering dependencies.

1.1. *Background*

We repeat the simpler version of RYS in [A.Mani (2012b)], for more details see [A.Mani (2012a); A.Mani (2012b); A.Mani (2013a)].

Definition 1.1. *The* language of RYS $\mathfrak{L}$ *will consist of*

- *Symbols for binary predicates* $\mathbf{P}, =, \stackrel{e}{=}$,
- *Binary partial operations:* $\oplus, \cdot$,
- *Unary operations* $(l_i)_1^n$, $(u_i)_1^n$, $\sim$ *and 0-place operations* 1 *(*0 *being optional).*
- *Existence equations* $t \stackrel{e}{=} s$ *are pairs of terms* s, t *and will be treated as atomic formulas (as in [Burmeister (2002)]). We will not make use of them in formal computations in this paper.*
- *Optional unary predicate or function* γ *for indicating granules (if we want to deal with inner RYS). It will not be included by default. We sometimes use* superfluous brackets *on initial quantifiers to aid readability (they should be dropped for interpretation as schema) and omit initial quantifiers in models.*

The axioms of *RYS* $\mathcal{A}$ *will be the basic equality and existence equality [Burmeister (2002)] axioms and the following:*

A1 $(\forall x)\mathbf{P}xx$.
A2 $(\forall xy)(\mathbf{P}xy,\ \mathbf{P}yx \longrightarrow x = y)$.
A3 *For each* i, $(\forall xy)(\mathbf{P}xy \longrightarrow \mathbf{P}x^{l_i}y^{l_i},\ \mathbf{P}x^{u_i}y^{u_i})$.
A4 *For each* i, $(\forall x)\mathbf{P}x^{l_i}x,\ \mathbf{P}xx^{u_i}$.
A5 *For each* i, $(\forall x)(\mathbf{P}x^{u_i}x^{l_i} \longrightarrow x = x^{l_i} = x^{u_i})$.

The partial operations $\oplus, \odot, \ominus$ *shall satisfy and the derived operations* $\mathbf{O}, \mathbb{P}, \mathbb{X}, \mathbb{O}$

will be assumed to be defined uniquely as follows:

$$\mathbb{P}xy \leftrightarrow \mathbf{P}xy \wedge \neg \mathbf{P}yx \tag{Proper Part}$$

$$\mathbb{X}xy \leftrightarrow \mathbf{O}xy \wedge \neg \mathbf{P}xy, \tag{Overcross}$$

$$\mathbb{O}xy \leftrightarrow \mathbb{X}xy \wedge \mathbb{X}yx, \tag{Proper Overlap}$$

$$(x \ominus y = z \rightarrow (\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \neg \mathbf{O}wy))) \tag{wDifference1}$$

$$(x \ominus y \stackrel{e}{=} x \ominus y, \Phi(x, y, z), \Phi(x, y, a) \longrightarrow x \ominus y = z = a), \tag{wDifference2}$$

where $\Phi(x, y, z)$ stands for $(\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \neg \mathbf{O}wy))$.

$$(x \oplus y = z \longrightarrow (\forall w)(\mathbf{O}wz \leftrightarrow (\mathbf{O}wx \vee \mathbf{O}wy))) \tag{Sum1}$$

$$(x \oplus y \stackrel{e}{=} x \oplus y, \Omega(x, y, z), \Omega(x, y, a) \longrightarrow x \oplus y = z = a), \tag{Sum2}$$

where $\Omega(x, y, z) \equiv (\forall w)(\mathbf{O}wz \leftrightarrow (\mathbf{O}wx \vee \mathbf{O}wy))$

$$(x \odot y = z \longrightarrow (\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \mathbf{P}wy))) \tag{Product1}$$

$$(x \odot y \stackrel{e}{=} x \odot y, \Pi(x, y, z), \Pi(x, y, a) \longrightarrow x \odot y = z = a), \tag{Product2}$$

where $\Pi(x, y, z) \equiv (\forall w)(\mathbf{P}wz \leftrightarrow (\mathbf{P}wx \wedge \mathbf{P}wy))$

$$x \oplus (y \oplus z) \stackrel{\omega*}{=} (x \oplus y) \oplus z. \tag{wAssociativity*}$$

$$x \oplus y \stackrel{\omega*}{=} y \oplus x;\ x \odot y \stackrel{\omega*}{=} y \odot x. \tag{wCommutativity}$$

* *For two terms* s, t, $t \stackrel{\omega*}{=} s$ *is an abbreviation for* $(s \stackrel{e}{=} s \rightarrow s \stackrel{e}{=} t) \wedge (t \stackrel{e}{=} t \rightarrow s \stackrel{e}{=} t)$.

Definition 1.2. *If all the operations are total, then a model of the theory of RYS* $\langle \mathfrak{L}, \mathcal{A} \rangle$ *will be an* $\mathfrak{L}$*-structure of the form*

$$\langle \underline{S}, \mathbf{P}, (l_i)_1^n, (u_i)_1^n, \oplus, \odot, \ominus, 1 \rangle$$

that satisfies the axioms $\mathcal{A}$. *In this case the existence equality will coincide with the usual equality and the the operations* $\oplus, \odot$ *would be defined by the axioms.*

If partial operations are present, then the formulation of a 2-valued model or a model will be more involved [A.Mani (2012a); Burmeister (2002)]. The intended concept of a *rough set* in a RYS is as a collection of some sense definite elements of the form $\{a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_r\}$ subject to a_is being 'part of' of some of the b_js and obtainable from the multiple approximation operators $\{l_i, u_i\}$. $\mathbf{P}xy$ can be read as 'x is a part of y' and for more details see the following section, though there is nothing preventing us from using it in the sense of *is a* alone. This happens when we require y to be a definite object or require y to be substitutable by definite objects unless x is the same as y. The elements of S may be approximable and/or exact objects (or collections thereof). The justification for using a non-transitive part-hood relation can be traced to various situations in which restrictive criteria operating on inclusion of attributes happen (see [A.Mani (2011); A.Mani (2012a)] for examples). A fragment of the axioms of set-theory compatible mereology in [Varzi (1996)] are assumed. This mereology is not ontologically compatible with Lesniewskian mereology and reformulations in the original language of ontology are needed for *comparisons*. In the Lesniewski-inspired mereology of Polkowski and Skowron [Polkowski and Skowron (1996)], the basic functor "part to a degree r", definitely assumes knowledge of the degree r and is responsible for the strong collectivizing properties of the class operator. The reason for avoiding such an approach, explained in [A.Mani (2012a)], primarily relate to the contamination problem that lays stress on the difference between the awareness at a rough semantic domain and the classical semantic domain.

The definitions of $\odot, \oplus, \ominus$ require two statements as they are intended to be interpreted as partial operations. Existence equalities of the form $s \stackrel{e}{=} s$ are intended to be interpreted as the term s *exists* or is defined. $\oplus$ is intended to be interpreted as a general aggregation operation, $\odot$ as a general commonality-extracting operation and $\ominus$ is a reserve operation for accommodating differences or complementation via 1. $\ominus$ may become completely redundant in applications to concept approximation in human learning contexts. In a model of RYS, the definition of sum, product and difference operations do not follow in general from the conditions on $\mathbf{P}$ in the above. But do so from the assumptions of closed extensional mereology ([A.Mani (2012a)]). Note that a minimalist concept of *admissible granulation* was defined in [A.Mani (2012a)] as those granulations satisfying the conditions WRA, LFU, LS.

We repeat the granular axioms for convenience below ($\mathcal{G}$ is supposed to be the collection of granules over the RYS model S and quantification is over S unless indicated otherwise):

$$\forall i,\ (\forall x)(\exists y_1, \ldots y_r \in \mathcal{G}) y_1 + y_2 + \ldots + y_r = x^{l_i}$$
$$\text{and} (\forall x)(\exists y_1, \ldots y_p \in \mathcal{G})\, y_1 + y_2 + \ldots + y_p = x^{u_i}, \quad \text{(Representability, RA)}$$
$$\forall i,\ (\forall x \exists y_1, \ldots y_r \in \mathcal{G}) t_i(y_1, y_2, \ldots y_r) = x^{l_i}$$
$$\text{and } (\forall x)(\exists y_1, \ldots y_r \in \mathcal{G})\, t_i(y_1, y_2, \ldots y_p) = x^{u_i}, \quad \text{(Weak RA, WRA)}$$
$$\exists i,\ (\forall x)(\exists y_1, \ldots y_r \in \mathcal{G}) y_1 + y_2 + \ldots + y_r = x^{l_i}$$
$$\text{and } (\forall x)(\exists y_1, \ldots y_p \in \mathcal{G})\, y_1 + y_2 + \ldots + y_p = x^{u_i}, \quad \text{(Sub RA)}$$
$$\forall i,\ (\forall x \exists y_1, \ldots y_r \in \mathcal{G}) t_i(y_1, y_2, \ldots y_r) = x^{l_i}$$
$$\text{and } (\forall x)(\exists y_1, \ldots y_r \in \mathcal{G}) t_i(y_1, y_2, \ldots y_p) = x^{u_i}, \quad \text{(Sub TRA, STRA)}$$
$$\forall i,\ (\forall x)(\exists y_1, \ldots y_r \in \mathcal{G}) y_1 + y_2 + \ldots + y_r = x^{l_i}, \quad \text{(Lower RA, LRA)}$$
$$\forall i,\ (\forall x)(\exists y_1, \ldots y_p \in \mathcal{G})\, y_1 + y_2 + \ldots + y_p = x^{u_i}, \quad \text{(Upper RA, URA)}$$
$$\exists i,\ (\forall x)(\exists y_1, \ldots y_r \in \mathcal{G})\, y_1 + y_2 + \ldots + y_r = x^{l_i}, \quad \text{(Lower SRA, LSRA)}$$
$$\exists i,\ (\forall x)(\exists y_1, \ldots y_p \in \mathcal{G})\, y_1 + y_2 + \ldots + y_p = x^{u_i}, \quad \text{(Upper SRA, USRA)}$$
$$\text{For each } i,\ (\forall y \in \mathcal{G})\, y^{l_i} = y^{u_i} = y, \quad \text{(Absolute Crispness,ACG)}$$
$$\exists i,\ (\forall y \in \mathcal{G}) y^{l_i} = y^{u_i} = y, \quad \text{(Sub Crispness*, SCG)}$$

Crispness Variants: LACG, UACG, LSCG, USCG will be defined as for representability, MER Variants:UMER, LSMER, USMER, IUMER can be defined by analogy in the following,

$$\forall i,\ (\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy,\ x^{l_i} = x^{u_i} = x \longrightarrow x = y),$$
(Mereological Atomicity,MER)

$$\exists i, (\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy,\ x^{l_i} = x^{u_i} = x \longrightarrow x = y)$$
(Sub MER,SMER (was 'weak MER' in [A.Mani (2011)]))

$$(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy, \bigwedge_i (x^{l_i} = x^{u_i} = x) \longrightarrow x = y), \quad \text{(Inward MER, IMER)}$$

$$\forall i, (\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy,\ x^{l_i} = x \longrightarrow x = y), \quad \text{(Lower MER, LMER)}$$

$$(\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}xy, \bigwedge_i (x^{l_i} = x) \longrightarrow x = y), \quad \text{(Inward LMER, ILMER)}$$

$$\forall i,\ (\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}yx \longrightarrow \mathbf{P}(y)(x^{l_i})), \quad \text{(Lower Stability, LS)}$$

$$\forall i, (\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{O}yx \longrightarrow \mathbf{P}yx^{u_i}), \quad \text{(Upper Stability, US)}$$

$$LS \,\&\, US, \quad \text{(Stability, ST)}$$

$$\exists i,\ (\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{P}yx \longrightarrow \mathbf{P}(y)(x^{l_i}))$$
(Sub LS, LSS (was 'LS' in [A.Mani (2011)]))

$$\exists i,\ (\forall y \in \mathcal{G})(\forall x \in S)(\mathbf{O}yx \longrightarrow \mathbf{P}(y)(x^{u_i}))$$
(Sub US, USS (was 'US' in [A.Mani (2011)]))

$$LSS \,\&\, USS, \quad \text{(Sub ST, SST)}$$

$$(\forall x,\ y \in \mathcal{G})\neg\mathbb{O}xy, \quad \text{(No Overlap, NO)}$$

$$\forall i,\ (\forall x,\ y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz,\ \mathbb{P}yz,\ z^{l_i} = z^{u_i} = z, \quad \text{(Full Underlap, FU)}$$

$$\forall i, (\forall x,\ y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz,\ \mathbb{P}yz,\ z^{l_i} = z, \quad \text{(LFU)}$$

$$\exists i, (\forall x,\ y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz,\ \mathbb{P}yz,\ z^{l_i} = z^{u_i} = z, \quad \text{(Sub FU, SFU)}$$

$$\exists i, (\forall x,\ y \in \mathcal{G})(\exists z \in S)\mathbb{P}xz,\ \mathbb{P}yz,\ z^{l_i} = z, \quad \text{(Sub LFU)}$$

$$\exists i, (\forall x, y \in \mathcal{G})(\mathbb{P}xz, \mathbb{P}yz,\ z^{l_i} = z^{u_i} = z,\ \mathbb{P}xb, \mathbb{P}yb,$$
$$b^{l_i} = b^{u_i} = b \longrightarrow z = b), \quad \text{(Unique Underlap, UU)}$$

$$(\forall x,\ y \in \mathcal{G})(\exists z \in \mathcal{G})\mathbf{P}(x \cdot y)(z), \quad \text{(Pre-similarity, PS)}$$

$$\forall i, (\forall x \in \mathcal{G})x^{l_i} = x^{l_i l_i}, \quad \text{(Lower Idempotence, LI)}$$

$$\forall i, (\forall x \in \mathcal{G})x^{u_i} = x^{u_i u_i}, \quad \text{(Upper Idempotence, UI)}$$

$$\forall i, (\forall x \in \mathcal{G})x^{u_i} = x^{u_i u_i},\ x^{l_i} = x^{l_i l_i}. \quad \text{(Idempotence, I)}$$

*In [A.Mani (2011)], this was termed 'weak crispness'.

Definition 1.3. *By the theory of* Classical RST-RYS*, we will mean a theory* $\mathfrak{Th}$ *of RYS in which* $\oplus, \odot, \ominus$ *correspond respectively to set* $\cup, \cap, \setminus$ *respectively.* S *is a power-set of some set* A*,* $1 = A$*, and the additional granular axioms RA, ACG, MER, FU, NO, PS, ST, I (see [A.Mani (2012a)]) hold. Idempotence (I) is* $\forall i,\ (\forall x \in$

$\mathcal{G})x^{u_i} = x^{u_i u_i}, x^{l_i} = x^{l_i l_i}$.

Theorem 1.4. *The theory of classical RST-RYS is well defined, is not categorical or* κ*-categorical (*κ *being a cardinal) and is consistent.*

Proof. For a fixed cardinality of S, we can define multiple non-isomorphic models of classical RST-RYS. By Thm 2 of [A.Mani (2012a)], we know that all of RA, ACG, MER, ST, FU, NO, PS hold, but UU does not. So the theory of classical RST-RYS is consistent. □

Any relation $\varphi : S_1 \longmapsto S_2$ will be taken to be a *correspondence* between the systems. though of course only those that are maps and /or preserve granularity or approximations in some sense would be of interest.

1.2. *Meaning*

The concept of meaning and its connection with semiotics can be formulated in different ways based on one's philosophical position. This can affect our computational model in severe ways. Our intended application contexts may be of a purely mechanical nature or an AI related context or a context for studying human behavior - such variation is also related to these theories of meaning. Hints on key aspects are provided below.

Our position is basically a form of dialectical materialism and assumes the essence of realism opposed to positivism. We presuppose

- ⋆ that the world exists, since we can experience, think of, represent the thoughts about, and act upon it,
- ⋆ that the existence of the world is independent of our experience, thought, expression, and our actions, and
- ⋆ that the world is such as we experience it through perception and action, and, with consistency (and without contradiction), can think and express those thoughts about it.

Every rational concept in my view is grounded on components emerging from perception that become amenable to logical thought processes aimed at higher levels of clarity. This view is opposed to positivist perspectives of seeing interpretations as facts.

In a few realist perspectives it can be argued that a key problem with Frege's approach to analysis of thought through the analysis of language is the isolation of language and thought from other kinds of action (interaction) leading to lack of coherence between language and thought. The point holds inspite of concepts like those of *speech acts* and *language use* - because subjective thought is rejected as *mentalism*. Ideas of hierarchical integration of subjective and intersubjective thought are also incompatible with the Fregean approach (Admittedly there are plenty of vague components in this view of Fregean approach). For more on our approach to handling subjectivity or alleged subjectivity, see the following subsection.

Many Researchers in AI [Yao (2004); Kazenzadeh *et al.* (2013); Pagliani and Chakraborty (2007); Pagliani and Chakraborty (2008)] tend to oversimplify Frege's conception (which presupposes an external world) of *intension* and *extension* for application. Thus concepts of human knowledge are taken to consist of an intensional part and an extensional part. The *intension* of a concept is the collection of properties or attributes that hold for the set of objects to which the concept applies. The *extension* is to consist of actual examples of the objects. There are differences in the extent to which interpretation of symbolic expression is distributed between the *intension* and *extension*.

We can see an expression of the form $f(x) = x^3$ as a string of symbols. It can be interpreted in many models, but we can specify that only an explicit specification as sets of ordered pairs of real numbers amounts to an extension of the formula. But compact expressions of models can also be seen as extensions (though not in a Fregean sense). Fregean concepts of meaning relate to sense and reference. Some circularity with knowability is natural.

Following [Haukioja (2009)], let us call a fact c-knowable when it is such that one can know it merely on the basis of being competent with the relevant concepts. c-knowable propositions are á priori if genuine knowledge is possible without sensory experience. c-knowledge may be interpreted as conceptual knowledge that is justifiable by empirical conditions if that is not necessarily true. Natural kind terms constitute a class of general terms and include both mass terms, like *water*, and certain sortal terms, like *river* and *girl*. According to Kripke, natural kind terms are rigid designators. We agree with the claim [Haukioja (2009)] that conditional conjunctive specifications of reference-fixers must be c-knowable if intuition-driven thought experimentation is the proper tool for doing the theory of reference for natural kind terms and that it is at best misleading to claim that thought experiment can show that reference-fixers are not c-knowable.

A commitment to semantic domains in rough theories may be seen to be connected with an externalist view of meaning or content (that is the view that the semantic content of a *concept* depends on the nature of the external world) . But externalists do not care about the domains and may therefore be seen to be vague when they are talking about *concepts*. In fact this has been implicit in the present author's position from both philosophical and technical points of view - rough concepts are specific to the domain in which we are talking about them. Existence of correspondences between concepts in different semantic domains or even a universal classical semantic domain does not mean that the concepts are the same. Externalism is like neo-Platonism in presupposing abstract concepts under which objects may fall in some world. This *concept* differs from our rough set theoretical one.

The above and the position on dualism is very important for the computing process as the formalism of the context can be affected by these. In fact, different positions on the non-dualism vs dualism debate lead to distinct computing processes that they may relate to.

1.3. *Handling Subjectivity*

We can dismiss subjectivist realizations as something not too different from pure relativism in formal approaches. But this would be an oversimplification with many negative consequences. From a pragmatic perspective, we know too well that subjective features are common in application contexts in the semantic domain in question. Further these have differences between themselves from universal points of view. Some references are [Stern (2004); Keefe (2000)].

When we talk about correspondences between RYS, then a good understanding of the nature of variation between corresponding properties or objects can have deep significance in formal semantics and computation. Some aspects have been explored by the present author in [A.Mani (2013a)]. One of concepts is that of being small enough when we have no real measures associated. Thus the relevance of exploring the nature of the subjective is justified. This approach is essentially opposed to subjectivism that refer to approaches subscribing to universal subjectivitity with related presuppositions of a universal nature.

Aesthetic subjectivism is sometimes supposed to be commonsensical and philosophically plausible - though views opposing this are also common mainly because it seems to lead to *excesses of tolerance* and because it seems grounded in a lack of commitment to reason on finer properties. For Hume, *All sentiment is right* in aesthetic matters and aesthetic judgements are supposed to be un-amenable to reason as aesthetic qualities do not inher in objects nor do they simply report the subjects experiences. A mathematical schemata of Hume's view would be - an object appears to have an aesthetic quality Q if it generates a correct response $\mathsf{R}(\mathsf{Q})$ in the subject (and possible responses must be classifiable as correct or incorrect).

For Hume, a sweet is delicious because of the contingent facts that it tastes similarly for many people as they have similar sensibilities and that the sweet was meant to please. Differences in judgement occur due to differences in aesthetic judgements or the sense of taste.

Kant, in contrast, rejects Hume's view because the central question of "what a correct aesthetic judgement ought to be?" is not addressed. For Kant, we have *pure sense perceptions* in us that we share with other people. When we succeed in removing all the noise from the data relative that sense perception, then we can deduce the essence of the universal aesthetic in the object that lies beyond the empirical world. This metaphysical position is problematic in materialistic perspectives and naturally leads to disputes about what really is *pure sense perception*.

Even if we assume that aesthetic subjectivism is real then the natural questions would be about

- ⋆ the extent of such subjectivism,
- ⋆ measurability of such subjectivism from a higher order perspective, and

- ⋆ comparability of such subjectivisms.

We do not see much merit in Wittengstein's position that aesthetic judgements are like gestures and exclamations [Wittgenstein (1978)] - their role being to suggest an experience and not really to say much about the object as such. This aspect is explored in Chapter-6 of [Wollheim (1980)] for example.

Many authors using the procedures of exact reasoning and logics in limited domains of discourse have claimed wild negative features of aesthetic reasoning. Matters are further compounded by pluralist terms like *beauty*. We do not see much formal merit in these claims simply because we can potentially handle all of the following in a suitable semantic domain with

- ⋆ consistency,
- ⋆ logical rules and consequence accommodating vagueness and
- ⋆ generalizations or inductive reasoning.

We have discussed aesthetic subjectivism instead of plain subjectivism in general because the former has more diversity and provides insights for handling subjectivity. Of course researchers in vagueness and AI often use/accommodate subjective judgements in various situations. One of the concerns of the contamination problem [A.Mani (2012a)], is the use of low quality numeric measures for handling subjectivity.

1.4. *Theories of Vagueness*

We are not committing ourselves to a specific theory of vagueness like supervaluationism [Keefe (2000); Schiffer (2010)] - the main reasons, as implied in the present author's earlier papers [A.Mani (2012a)], is that our scope of possible problems is wider. Paradoxes like the sorities paradox refer to a limited set of semantic domains and the vague predicates used have substantial numeric import. Our concerns are also about possible origin vagueness in knowledge.

When we speak of vague predicates their meaning or reference need not be exactly fixed relative all interpreters. But we can always formulate a semantic domain to cover them all - this is a nice category theoretic property to have at all times. This in conjunction with our approach to subjectivity and the contamination problem roughly specifies our position.

1.5. *Proto-Transitive Rough Sets*

Proto-transitivity is one of the infinite number of generalizations of transitivity. Proto-transitive approximation spaces (PRAX) have been introduced by the present author in [A.Mani (2013c)] and the structure of definite objects and knowledge interpretation investigated therein. It is a relatively hard structure from a semantic perspective as the representation of rough objects is involved [A.Mani (2013c)]. Though as many as six different semantic approaches have already been developed by the present author, there is scope for further enhancement. The sixth in [A.Mani (2013b)] is based on correspondences and approximation of the prototransitive relation of a PRAX. The approach works because Nelson algebras form a semantics corresponding to the approximate transitive relation. We prove an internalization process in this paper through the dependency relation.

If R is a relation on a set S, then R can be approximated by a wide variety of partial/quasi-order relations in both classical and rough set perspective [Janicki (2011)]. Though the methods are essentially equivalent for binary relations, the latter method is more general. When the relation R satisfies proto-transitivity, then many new properties emerge. This aspect is developed in some detail in the present paper.

When R is a quasi-order relation, then a semantics for the set of ordered pairs of lower and upper approximations $\{(A^l, A^u);\, A \subseteq S\}$ has recently been developed in [Jarvinen and Radeleczki (2011); Jarvinen *et al.* (2012)]. Though such a set of ordered pairs of lower and upper approximations are not rough objects in the PRAX context, we can use the approximations for an additional semantic approach to it. We prove in [A.Mani (2013b)] that differences of consequent lower and upper approximations suggest partial structures for *measuring* structured deviation. The methods developed in that paper are also relevant for studying correspondences between the different semantics [A.Mani (2012b); A.Mani (2013a)].

Rough objects as explained in [A.Mani (2011); A.Mani (2012a)] are collections of objects in a classical domain (Meta-C) that appear to be indistinguishable among themselves in another rough semantic domain (Meta-R). But their representation in most RSTs in purely order theoretic terms is not known. For PRAX, this is solved in [A.Mani (2013c)]. Rough objects in a PRAX need not correspond to intervals of the form $]a, b[$ with the definite object b covering (in the ordered set of definite objects) the definite object a.

Definition 1.5. *A binary relation* R *on a set* S *is said to be* weakly-transitive, transitive or proto-transitive *respectively on* S *if and only if* S *satisfies*

- ⋆ *If whenever* Rxy, Ryz *and* $x \neq y \neq z$ *holds, then* Rxz. *(i.e.* $(R \circ R) \setminus \Delta_S \subseteq R$ *(where* $\circ$ *is relation composition) , or*
- ⋆ *whenever* $Rxy \,\&\, Ryz$ *holds then* Rxz *(i.e.* $(R \circ R) \subseteq R$*), or*
- ⋆ *Whenever* Rxy, Ryz, Ryx, Rzy *and* $x \neq y \neq z$ *holds, then* Rxz *follows, respectively. Proto-transitivity of* R *is equivalent to* $R \cap R^{-1} = \tau(R)$ *being weakly transitive.*

Note that *weak transitivity* of [Chajda and Haviar (1991)] is *proto-transitivity* here. $Ref(S)$, $w\tau(S)$, $p\tau(S)$, $EQ(S)$ will respectively denote the set of reflexive, weakly transitive, proto transitive, and equivalence relations on the set S respectively. We can prove $w\tau(S) \subseteq p\tau(S)$ and $\forall R \in Ref(S)(R \in p\tau(S) \leftrightarrow \tau(R) \in EQ(S))$.

2. Mereo-Ontological Aspects of RYS

We look at part-hood from a practical perspective in a RYS so that possible ontologies become less dense. In a practical perspective of course S need not be a set, but many situations fall within such a context. Even under such a situation, the possible ontologies can be very varied.

$\underline{S}$ may be a power-set of a set, a subset thereof, a subset of a power set of a power-set. There is nothing that causes restrictions on the ontological types of objects or collections of objects or predicates involved in these considerations. Part-hood may be seen as an improper generalization of the concept of subset and only some of these may be seen as the result of modifiers acting on usual set-theoretic inclusion. Some of the possibilities in the modifier based perspective are as follows:

Basic Inclusion: $A \subseteq B \subseteq C$.

Representable Rough Inclusion: We refine the previous terminology for all the confusion that it might have or seems to have caused. In all that follows, RYS corresponds to a model of RYS.

Definition 2.1. *Let $\Phi(x, y)$ be a formula obtained as a conjunction of formulas of the form $x^\alpha \subseteq y^\alpha$ for some approximation operators α in the signature of the RYS model S. A will be said to be* representably roughly included *in B relative formula Φ if and only if $\Phi(A, B)$ is satisfied in the RYS S.*

Definition 2.2. *Let $\Psi(x, y)$ be a formula obtained as a conjunction or disjunction of formulas of the form $x^\alpha \subseteq y^\alpha$ for some approximation operators α in the signature of the RYS model S. A will be said to be* weak representably roughly included *in B relative formula Ψ if and only if $\Psi(A, B)$ is satisfied in the S.*

Rough Inclusion: This definition is very broad, but discludes many property based orderings and may not be transitive.

Definition 2.3. *Let $\Xi(x, y)$ be a formula obtained as a Boolean combination of formulas of the form $x^\alpha \subseteq y^\beta$ for some approximation operators α, β in the signature of the RYS S. A will be said to be* roughly included *in B relative formula Ξ if and only if $\Xi(A, B)$ is satisfied in the RYS S.*

Modifiers Over Inclusion: For any two elements $x, y \in S$, if we define $\mathbf{P}xy$ if and only if $\phi(x) \,\&\, \phi(y) \,\&\, x \subseteq y$ for some property ϕ, then $\mathbf{P}$ can be said to be defined through the action of modifiers on inclusion. In general this part of the definition of $\mathbf{P}$ can be external to the RYS and need not be definable within it. This permits a wide variety of possibilities. A key question would be a description of the objects excluded by way of restrictions on types. This is taken up in subsection 2.1.

Incoherent Part-hoods: If we start with some partial/quasi order on $\underline{S}$, then of the many situations in which it may happen that $\mathbf{P}xy \,\&\, y < x \,\&\, x \neq y \,\&\, x \not\leq y$ or $\mathbf{P}xy \,\&\, y < x \,\&\, x \neq y$, the cases in which part-hood is defined through approximations will be of much interest. If we define $\mathbf{P}xy$ if and only if y is an approximation of x or $x = y$, then in even from situations of classical RST, we have an intricate part-hood that expresses an aspect of contamination avoidance. The reasoning in the context may be that *any approximation of a vague object is preferable over the vague object*.

ISA : A predicate of the form *is a* is not a problem in a RYS and is one natural perspective for speaking of rough approximations. The important thing here is to determine the extent of admissibility in relation to possible ontologies of the type of object involved. For example, a statue is an amount of matter, a lower approximation of *a statue* is *artistic quality X* (say) and there is nothing which prevents us from excluding this lower approximation being a similarly constructed lower approximation of *amount of matter*.

One important question can be that of reduction of all kinds of part-whole relations to the RYS perspective. In my opinion this kind of question misses the point as every such context can be formulated within the perspective and with some concept of approximations. The real question would be about the amenability of the approximations used rather than anything else. In the version restricted to the set theoretical perspective, the axiom A2 is restrictive in requiring part-hood relations to be antisymmetric. It is easy to violate this even in rough reasoning contexts:

Example:

In a sense, a river is part of the ocean it drains into and in a ecological sense, the ocean is part of the river. The two senses have sufficient commonality in them to permit a single part-hood relation for formalization. And the ocean is distinct from the river. This situation is because we do not directly impose sufficient constraints connecting the type of objects and part-hood. The natural way to solve this is by permitting more equalities and has been pointed out earlier in my earlier paper [A.Mani (2012a)]. But the argument between the approach of selecting/defining a better part-hood and selecting better semantic domains is nowhere near closed.

The well known approach [Winston *et al.* (1987)] to classification of part whole relations actually depends on or is rather based on the type of *objects* involved (I use *objects* in a very wide not necessarily mathematically representable sense). This point of view is shared by others including [Vieu (2007); Vieu and Aurnague (2007)]. Standard examples for the relations Component – Integral Object (CIO) , Member –Collection (MeC) , Portion – Mass (PoM), Stuff – Object (St0b), Feature – Activity (FeA) and Place – Area (PlA) are respectively *Handle – Cup, Tree – Forest, Slice – Pie, Steel – Bicycle, Paying – Shopping, and Oasis – Desert* respectively.

If one girl is in relationship with another girl then each of the girls is 'part of' the relationship. The consequence may be written as

$$\mathbf{P}[\mathsf{Girl1}][\mathsf{Rel}]\,\&\,\mathbf{P}[\mathsf{Girl2}][\mathsf{Rel}].$$

The concept of relationship in this case is obviously not independent of the context. Classification in accordance with the scheme of [Winston *et al.* (1987)] depends on how we interpret the dependence. If we are talking about a specific kind of relationship and not *the relationship*, then it is difficult to specify the category, but if we are talking about *the relationship*, then the closest category is that of [Component, Integral Object]. So vague entities do not cohere with the above classification. *In sharp contrast, a* RYS *can easily handle this kind of scenario.*

In the literature on ontology, part-whole relations are also classified on the basis of vaguely defined properties of the part-hood relation itself between the entities in the context (apart from those based on types of entities). This classification invariably corresponds to extraneous aspects and associated ordering relations between the respective properties of the part and whole. One common physicalist approach is to classify as per the properties of parts being (μ- is used to distinguish from the mathematical notions expressed by the same terms) μ-*functional* or μ-*homemerous* or μ-*separable* or otherwise relative the whole.

μ- *Functionality* corresponds to the parts being restricted in their spatial or temporal position relative the whole. Thus **handle of a cup** is functional part of the **cup**.

μ-homeomerosity corresponds to relevant parts and wholes being made of the similar kind of thing. An example would be that of a **slice** of **pie**. A tree in contrast is said to be a non homeomerous part of a forest. Relative our RYS approach, this refers to a specific kind of mensurable granularity. The mensurability aspect is relevant at specific levels of granularity.

μ-separability refers to separability of the part from the whole in principle (so that they have separate existence). Thus **handle of a cup** is a separable part of the **cup**, while **steel of a bicycle** is not separable from the **bicycle**. Here again we have some connection with levels of granularity and axioms too. At one molecular level of granularity steel of bicycle is separable from the bicycle, but at a macro level the aforesaid is true.

By changing the philosophical assumptions we can contradict table below (1) that in my opinion expresses a philosophical standpoint on the existence of objects and their interrelations. The evidence for such a conclusion is in the common examples cited in the literature [Vieu and Aurnague (2007)]. It follows that Winston's taxonomy of relations is too vague and does not accommodate granularity to any substantial extent. In the table **F, H, S** respectively refer to functionality, homeomerosity and separability respectively.

Table 1. Objects and Relations

Relation	**F**	**H**	**S**
CIO	Yes	No	Yes
MeC	No	No	Yes
PoM	No	Yes	Yes
StOb	No	No	No
FeA	Yes	No	No
PlA	No	Yes	No

Many of the formal approaches in ontology proceed from *some primitive classification* aggregate to *more evolved ones*. In [Masolo *et al.* (2004)], the basic predicate for *classified as at a temporal instant* starts from a **depends on** predicate that again relates to existence at temporal instants and equalities. $\mathbf{E}xt$ is read as 'x exists at time t' and a modal reading of the predicate 'Depends On' is via

$$\begin{aligned}\Delta(x,y) \equiv\ &\Box(\forall t)(\mathbf{E}xt \to \mathbf{E}yt)\&\neg(x=y)\\ &\&\Diamond((\exists t\,\mathbf{E}xt)\&\neg\Box\forall t\,\mathbf{E}yt)\end{aligned}$$

where $\Box$, $\Diamond$ correspond to 'necessity' and 'possibility' modalities. Further a key axiom is $\mathsf{CF}(x, X, t) \rightarrow \mathbf{E}xt$, that is if x is classified as a X at time t, then x exists at time t. This approach is not compatible with various rough set theoretical scenarios mainly because of the $\mathbf{E}$ predicate when we are talking across semantic domains.

We are not avoiding predicates like *IS A* or *AS A* and neither do we have problems with basic predicates like $\Pi x\alpha t$ read x has property α at time t. But the status of existence of collection of such instances will require an admissibility criteria. It is implicit in [A.Mani (2012a)] that $\mathbf{P}$ can be taken as *IS A* in RYS with additional equalities.

We do not accept that every property is collectivizing (By this, we mean if H is a predicate then $\{x; Hx\}$ is in the collection of structures under consideration). If it were so, then we would also have the burden of collapse of ZF compatible set theory. Because of this the *classified as* predicate cannot always be regarded as a basic one. In practical situations we however use only collectivizing properties and related predicates. This will be the case in the main construction of this paper.

2.1. *Other Aspects of Ontologies*

The biggest problem of relating to many of the ontological approaches is their choice of linguistic domains of discourse and not a transformed semantic variant of it as in RST. The association of properties/attributes to objects is considered in some detail in many ontological approaches. Some aspects of this have already been considered above.

The idea of *functionality* as used in the literature on ontology [Vieu and Aurnague (2007)] is intended to give a good account of how and when parts have a role to play in wholes. In general RST, most accounts have consistently provided a lower level structured point of view and philosophy underlying functionality is likely to be interpreted as being pointless by many. This is so because functionality is based not on *properties of entities* but on *properties of entities that are associated to the lexical type (as in [Vieu and Aurnague (2007)]) of the entity.* However any such restriction amounts to computing approximations from a different view of properties and attributes would be amenable from the RYS perspective. The important question then would be about the axiomatic properties of granules [A.Mani (2012a)].

In [Collins and Varzi (2000)], the authors say that if a predicate generates the sorities paradox, then it is vague. To be precise, it should be said to be vague with respect to sustaining the induction argument in the context.

Ideas of rough sets as ontological information systems and information quanta [Pagliani and Chakraborty (2007); Wolski (2009)] are naturally relevant and have supplementary value in the formalization.

3. Rough Dependence

In this section, we introduce a concept of *rough dependence* in general rough set theory. By the term *rough dependence*, we intend to capture the relation between two objects (crisp or rough) that have some representable rough objects in common. There is no process for similarity with the concept *mutual exclusivity* of probability theory and in rough sets we are actually handling evolution without regard to temporality. We would like to eventually analyze the extent to which ontology of not-necessarily-rough origin can be integrated in a seamless way. But in this paper we will introduce basic concepts, compare them with probabilistic concepts and look at the semantic value of introduced functions and predicates.

Overall the following problems are basic and relevant for use in semantics:

- ⋆ Which concepts of rough dependence provide for an adequate semantics of rough objects in the PRAX context?
- ⋆ More generally how does this relation vary over other RSTs?
- ⋆ Characterize the connection between granularity and rough dependence?

By *relation based RST* we mean rough theories originating from generalized approximation spaces of the form $U = \langle \underline{U}, R \rangle$, with $\underline{U}$ being a set and R being any binary relation on $\underline{U}$. If instead of a relation we start from a cover of the set, then we will refer to the rough theory as a *cover based RST*.

Definition 3.1. *The* $\tau\nu$-infimal degree of dependence $\beta_{i\tau\nu}$ *of* A *on* B *will be defined as*

$$\beta_{i\tau\nu}(A, B) = \inf_{\nu(S)} \oplus\{C : C \in \tau(S) \,\&\, \mathbf{P}CA \,\&\, \mathbf{P}CB\}.$$

Here the infimum means the largest $\nu(S)$ *element contained in the aggregation.*

The $\tau\nu$-supremal degree of dependence $\beta_{s\tau\nu}$ *of* A *on* B *will be defined as*

$$\beta_{s\tau\nu}(A, B) = \sup_{\nu(S)} \oplus\{C : C \in \tau(S) \,\&\, \mathbf{P}CA \,\&\, \mathbf{P}CB\}.$$

Here the supremum means the least $\nu(S)$ *element containing the sets.*

The definition extends to RYS *in a natural way.*

Note that all of the definitions do not use real-valued rough measures and the cardinality of sets in accord with one of the principles of avoiding contamination. The ideas of dependence are more closely related to certain semantic operations in classical RST. But these were never seen to be of much interest. The connections with probability theories has been part of a number of papers including [Pawlak (1991); Pawlak (2004); Pawlak (2002); Slezak (2006); Yao (2008)], however neither dependence nor independence have received sufficient attention. This is the case with other papers on entropy. It should be noted that the idea of independence in statistics is seen in relation to probabilistic approaches, but dependence has largely not been given much importance in applications.

The positive region of a set X is X^l, while the negative region is X^{uc} – this region is independent from x in the sense of attributes being distinct, but not in the sense of derivability or inference by way of rules. When we talk of dependence or independence of a set relative another, then a basic question would be about possible balance between the two meta principles of independence in the rough theory and the relation to the granular concepts of independence.

Definition 3.2. *Two elements* x, y *in a RBRST or CBRST* S *will be said to be* PN-independent $I_{PN}(xy)$ *if and only if*

$$x^l \subseteq y^{uc} \,\&\, y^l \subseteq x^{uc}.$$

and two elements x, y *in a RBRST or CBRST* S *will be said to be* PN-dependent $\sigma_{PN}(xy)$ *if and only if*

$$x^l \nsubseteq y^{uc} \,\&\, y^l \nsubseteq x^{uc}.$$

Theorem 3.3. *Over the RYS corresponding to classical RST, we have the following properties of dependence degrees when* $\tau(S) = \mathcal{G}(S)$ *- the granulation of* S *and* $\nu(S) = \delta_l(S)$ *- the set of lower definite elements. We omit the subscripts* $\tau\nu$ *and braces in* $\beta_{i\tau\nu}(x, y)$ *in the following:*

(1) $\beta_i xy = x^l \cap y^l = \beta_s xy$ *(subscripts* i, s *on* β *can therefore be omitted).*
(2) $\beta xx = x^l$.
(3) $\beta xy = \beta yx$.
(4) $\beta(\beta xy)x = \beta xy$.
(5) $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$.
(6) $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$.
(7) $\beta xy = \beta x^l y^l = \beta x y^l$.
(8) $\beta 0x = 0;\ \beta x1 = x^l$.
(9) $(\mathbf{P}xy \longrightarrow \beta xy = x^l)$.

We prove this in the next section.

Theorem 3.4. *For classical RST, a semantics over the classical semantic domain can be formulated with no reference to lower and upper approximation operators using the operations* $\cap$, c, β *on the power-set of* S*,* S *being an approximation space.*

Proof. We have already shown that l is representable in terms of β. So the the result follows. □

3.1. *Dependence in PRAX*

When we set $\nu(S) = \delta_l(S)$ and $\tau(S) = \mathcal{G}(S)$ - the successor neighborhood granulation, then the situation in PRAX contexts is similar, but we cannot define u from l and complementation. However when we set $\nu(S) = \delta_u(S)$, then the situation is very different.

Theorem 3.5. *Over the RYS corresponding to PRAX with* $\mathbf{P} = \subseteq$, $\oplus = \cup$ *and* $\odot = \cap$, *we have the following properties of dependence degrees when* $\tau(S) = \mathcal{S}$ *- the granulation of* S *and* $\nu(S) = \delta_l(S)$ *- the set of lower definite elements. In fact this holds in any reflexive RBRST. We omit the subscripts* $\tau\nu$ *and braces in* $\beta_{i\tau\nu}(x, y)$ *in the following:*

(1) $\beta_i xy = x^l \cap y^l = \beta_s xy$ *(subscripts* i, s *on* β *can therefore be omitted).*
(2) $\beta xx = x^l$; $\beta xy = \beta yx$.
(3) $(x \odot y = 0 \longrightarrow \beta_i xy = 0)$, *but the converse is false.*
(4) $\beta(\beta xy)x = \beta xy$.
(5) $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$.
(6) $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$.
(7) $\beta xy = \beta x^l y^l = \beta xy^l$.
(8) $\beta 0x = 0$; $\beta x1 = x^l$.
(9) $(\mathbf{P}xy \longrightarrow \beta xy = x^l)$.

Proof.

(1) $\beta_i xy$ is the union of the collection of successor neighborhoods generated by elements x and y that are included in both of them. So $\beta_i xy = x^l \cap y^l = \beta_s xy$.
(2) $\beta xx = x^l$; $\beta xy = \beta yx$. is obvious
(3) If $(x \odot y = 0$, then x and y have no elements in common and cannot have common successor neighborhoods. If $\beta_i xy = 0$, then x, y have no common successor neighborhoods, but can still have common elements. So the statement follows.
(4) $\beta xy \subseteq x^l \subseteq x$ by the first statement. So $\beta(\beta xy)x = \beta xy$.
(5) $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$ follows by monotonicity.
(6) If $\mathbf{P}y^l z$ is the same thing as $y^l \subseteq z$. $\beta xy = x^l \cap y^l$ and $\beta xz = x^l \cap z^l$ by the first statement. So we have $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$.
(7) $\beta xy = \beta x^l y^l = \beta xy^l$ holds because l is an idempotent operation in a PRAX [A.Mani (2013c)].
(8) Rest of the statements are obvious. □

Even though the properties are similar for reflexive RBRST when $\nu(S) = \delta_l(S)$ and $\tau(S) = \mathcal{G}(S)$, there are key differences that can be characterized in terms of special sets.

- $\beta xy = z$ if and only if $(\forall a \in z)(\exists b \in z)\, a \in [z] \subseteq x \cap y$.

- So we can select a minimal $K_z \subseteq z$ satisfying $(\forall a \in z)(\exists b \in K_z)\, a \in [b] \subseteq x$ and $(\forall e \in K_z)\,[e] \subseteq x \cap y$. Minimality being with respect to the inclusion order.
- Let $\mathcal{P}_z$ be the collection of all such K_z and let $\mathcal{B}_z$ be the subcollection of $\mathcal{P}_z$ satisfying the condition: if $K \in \mathcal{B}_z$ then $(\forall a \in K)(\forall b \in [a])(\exists J \in \mathcal{B}_z)\, b \in J$. $\mathcal{P}_z$ will be called the local basis and $\mathcal{B}_z$, the local super basis of z.

Proposition 3.6. *For classical RST* $(\forall z)\, \mathcal{B}_z = \mathcal{P}_z$ *and conversely.*

Theorem 3.7. *In the context of 3.5, if we set* $\nu(S) = \delta_u(S)$ *and* $\tau(S)$ *is as before, then we have (by* βxy*, we mean* $\beta_i xy$*)*

(1) $\mathbf{P}(\beta xy)(\beta_{i\delta_l(S)}xy)$,
(2) $\mathbf{P}(\beta xx)(x^l)$*;* $\beta xy = \beta yx$.
(3) $(x \odot y = 0 \longrightarrow \beta_i xy = 0)$*, but the converse is false.*
(4) $\beta(\beta xy)x = \beta xy$.
(5) $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$.
(6) $(\mathbf{P}y^l z \longrightarrow \mathbf{P}(\beta xy)(\beta xz))$.
(7) $\beta xy = \beta x^l y^l$; $\mathbf{P}(\beta xy^l)(\beta x^u y^u)$.
(8) $\beta 0x = 0$; $\mathbf{P}(\beta x1)(x^l)$.
(9) $(\mathbf{P}xy \longrightarrow \mathbf{P}(\beta zx)(\beta zy))$
(10) $(\beta xy)^l = \beta xy$.

Proof.

(1) By definition $\beta_{i\tau\nu}(A, B) = \inf_{\nu(S)} \oplus\{C : C \in \tau(S) \,\&\, \mathbf{P}CA \,\&\, \mathbf{P}CB\}$, so βxy is the greatest upper definite set contained in the union of common successor neighborhoods included in x and y. So it is necessarily a subset of $x^l \cap y^l$. In a PRAX, u is not idempotent and in general $x^u \subseteq x^{uu}$ ([A.Mani (2013c)]). So $\mathbf{P}(\beta xy)(\beta_{i\delta_l(S)}xy)$.
(2) The statements $\mathbf{P}(\beta xx)(x^l)$ and $\beta xy = \beta yx$ follow from the above.
(3) The proof is similar to that of third statement of 3.5.
(4) In constructing $\beta(\beta xy)x$ from βxy, we are not searching for upper definite subsets strictly contained in the latter. So the property follows.
(5) $\mathbf{P}(\beta xy)(\beta x(y \oplus z))$ follows by monotonicity.
(6) Obvious from previous statements.
(7) Note that $\beta x^u y^u$ is a subset of $x^u \cap y^u$ and in general contains βxy.
(8) Is a special case of the first statement. 0 is the empty set and 1 is the top.
(9) Follows by monotonicity.
(10) Upper definite subsets are necessarily lower definite, so $(\beta xy)^l = \beta xy$. □

The main properties of PN-dependence is as below:

Theorem 3.8. *In the context of 3.5, all of the following hold (we drop the subscript 'PN' in* σ_{PN}*):*

(1) σxx.
(2) $(\sigma xy \leftrightarrow \sigma yx)$.
(3) In general, $\sigma xy \,\&\, \sigma zy$ *does not imply* σxz. *But* $\neg\sigma xz$ *is more likely if we assume a bit of frequentism.*
(4) In general, $\sigma xy \nrightarrow \sigma x^u y^u$ *and* $\sigma x^u y^u \nrightarrow \sigma xy$.
(5) $(x \cdot y = 0 \longrightarrow \neg\sigma xy)$.
(6) $(\mathbf{P}xy \longrightarrow \sigma xy)$.

Theorem 3.9. *In the context of 3.5, if* $\beta xy \neq 0$ *then* σxy*, but the converse need not hold. In classical* RST*, the converse holds as well.*

Proof. If $\beta xy \neq 0$, then it follows that $x^l \cap y^l \neq \emptyset$ under the assumptions. If we assume $x^l \subseteq y^{uc} \vee y^l \subseteq x^{uc}$, then in each of the three cases we have a contradiction. So the first part of the result follows.

In the classical case, if $x^l \subseteq y^{uc}$ is not empty, then it should be a union of successor neighborhoods and similarly for $y^l \subseteq x^{uc}$. These two parts should necessarily be common to x^l and y^l. So the converse holds for classical RST. The proof does not work for PRAX and we know why it does not succeed. □

4. Comparison with Dependence in Probabilistic Theories

Probability measures may not exist in the first place over any given collection of sets, so even CBRST is necessarily more general and the idea of mutual exclusivity is not the correct concept corresponding to rough dependence. The basic idea of probabilistic dependence is oriented because occurrence of an event can be favorable or unfavorable for another event. In standard versions of rough set theory this has no corresponding concept. The concept of dependence in probability is rarely considered in the literature. The version in [Dimitrov (2010)] uses a not-so intuitive valuation but is nevertheless useful. We abstract the subjective aspect of the valuation for comparison.

Among the different understandings of probabilistic causation, frequentism ([Hajek (2009)]) and the tendency to omit necessary conditions are particularly problematic in various soft computing situations. In avoiding real-valued rough measures, we are committing to avoid the excesses of frequentism in rough sets.

If $(X, \mathcal{S}, p)$ is a probability space with X being a set, $\mathcal{S}$ being a σ-algebra over X and p being a probability function (we can use a collection of probability functions and handle more complex notions of dependence in 'probability structures', but these add little to the comparison), then the most natural dependence function $\delta : \mathcal{S}^2 \longmapsto \mathfrak{R}$ is defined by

$$\partial(x, y) = p(x \cap y) - p(x) \cdot p(y)$$

This function satisfies a number of properties that can be used to characterize dependence. In the subjective probability domain where p takes value in a bounded partially ordered partial semi-ring or your favorite partially ordered algebra, we will need to replace δ with a pair of predicates. So orientation of dependence seems to be fundamental in general forms of probability theory as well.

Two events $x, y \in X$ are *mutually exclusive* if and only if $x \cap y \neq \emptyset$. This concept can be extended to countable sets of events in a natural way. Also it is worthwhile to modify the concept of mutual exclusivity as in following definition:

Definition 4.1. *Two events* x, y *will be said to be* weakly mutually exclusive *(WME) if and only if*

$$x \cap y \neq z \,\&\, p(z) = 0.$$

Most results of probability theory involving mutual exclusivity continue to hold with the weaker assumption of WME and importantly is a better (though artificial) concept for comparison with the situation for rough sets.

Definition 4.2. *In the above context, let*

- πxy *if and only if* $p(x) \cdot p(y) < p(x \cap y)$
- σxy *if and only if* $p(x \cap y) < p(x) \cdot p(y)$

Proposition 4.3. *All of the following hold in a probability space:*

- $\pi xy^c \leftrightarrow \sigma yx$
- $\pi xy \leftrightarrow \pi yx$
- $(x \cap y \neq \emptyset \longrightarrow (\pi xa \,\&\, \pi ya \longrightarrow \pi(x \cup y)a))$
- $(x \cap y \neq \emptyset \longrightarrow (\sigma xa \,\&\, \sigma ya \longrightarrow \sigma(x \cup y)a))$
- $(\emptyset \neq x \subseteq y \longrightarrow \pi xy)$
- $(x \cap y = \emptyset \longrightarrow \sigma xy)$

Instead of using the the function $\partial(x, y)$, we can use the relations π, σ,as the former lacks a comparable contamination-free counterpart in rough set theory and also has peculiar properties like $\partial(x, x) \in [0, 1/4]$.

Proposition 4.4. *In the probability space above* $0 \leq \partial(x, x) \leq 0.25$, $-0.25 \leq \partial(x, x^c) \leq 0$ *and* x, y *are independent implies* $\partial(x, y) = 0$, *but not conversely.*

Proof. The proof of the inequalities follow by a simple application of real analysis. □

So it follows that the interpretation of the function $\partial(x, y)$ as in [Dimitrov (2010)] is actually incomplete. It combines certainty of the event with dependence.

Even though we can speak of positive, negative and neutral regions corresponding to an arbitrary subset A of a RBRST or CBRST S, natural ideas of dependence do not correspond to the scenario in probability space. In fact,

Theorem 4.5. *Predicates having properties identical with those of* π *and* σ *cannot be defined in the context of 3.5.*

Proof of this and more general results will appear separately.

5. Further Directions and Remarks

Evolution-wise there are too many differences between probabilistic events and rough approximations. We think comparison between the two theories is always bound to be relative abstract concept frameworks that we identify to be interpreted over the two. In comparison to the relatively difficult contamination-free ones introduced in this paper, the numeric measure based ones (including the Bayes factor approach) lack a ontologically sound theoretical basis. Irrespective of this, dependence is useful in the semantics of PRAX and other rough sets.

Some of the problems and questions on the connection with probability-like theories that this research motivates are:

- How does PN-dependence relate to granularity in different rough theories?
- What is a natural concept of granulation compatible with PN-dependence?
- What is an abstract characterization of the concept framework mentioned above?

In this research we have explained the connections of applied ontology with the mereological approach in rough Y-systems. This paves the way for further work on specific types of granular ontology and of course for easier applicability of RYS in practice. We have also introduced concepts of contamination-free dependence in general rough sets and probability spaces, demonstrated basic properties in specific kinds of rough sets and have shown their general incompatibility.

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