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Uncertainty Analysis on Neutron Diffusion Equation Using Fuzzy α-cut Approach

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Neutron diffusion equation is being used to estimate the safety margin of a neutronic parameter. The assessment of safety margin requires variability and uncertainty analysis. When the variation of sensitive parameter is varying in an imprecise manner, the evaluation of safety margin is very difficult using probabilistic analysis. However, fuzzy set theory based approach may be used to estimate the safety margin associated with imprecise variation. In this paper, impact of imprecise variation of cross section on neutron multiplication factor by solving neutron diffusion equation has been demonstrated.

Keywords: Multiplication factor; fuzzy set theory; diffusion equation.

1. INTRODUCTION

In the engineered nuclear system for nuclear power production, nuclear energy is produced through controlled nuclear fission reaction [Lamarsh (1983)] which is a bombardment driven process that results from collision of neutron with a fissile elements e.g. ^{235}U, ^{233}U and ^{239}Pu. Each fission reaction yields two to three neutrons that induced fission reaction in a chain manner which is known as fission chain reaction. Depending on the design of neutron multiplying system composed of fuel,

*Scientific Officer, Nuclear Safety Analysis Division, Atomic Energy Regulatory Board, Anushaktinagar, Mumbai-400094, INDIA moderator, reflector and control materials, the neutron yield can be increasing, decreasing or at a constant value with the progress of chain reaction. Based on the neutron yield profile, the system can be defined as super critical, subcritical and critical configuration. The characterization of neutron yield profile is being done with the measure of neutron multiplication factor. The neutron multiplication factor is defined as the ratio of neutron production in the current generation to the previous generation. The various processes are involved to control neutron population in a neutron multiplication system such as fission with thermal neutron, fission with fast neutron, absorption, resonance effect, thermal utilization, and leakage due to the finite system [Duderstadt and Hamilton (1976)]. The characterizations of neutronic properties such as neutron flux and neutron multiplication factor [Lamarsh (1983)], etc are important from safety point of view. The safety of a neutron multiplying system is ensured by safety margin, which is the deviation of estimated value from its safety limit provided by regulatory body. Determination of safety margin requires the knowledge of variability and sensitivity of uncertain parameters. Various methods are available to assess the variability, sensitivity and uncertainty analysis [Cacuci (2003)]. If the uncertain parameter is varied randomly following a particular distribution function, then uncertainty of the safety parameter can be estimated using probabilistic studies [Liu (2007)] to quantify the aleatory uncertainty [Bera S. *et al.* (2014)]. However, imprecise variation of sensitive parameter due to the lack of knowledge is very difficult to model using probabilistic methodologies. In a neutron multiplication system, the cross section that represents the probability for occurrence of a particular type of nuclear reaction such as fission, absorption, scattering, etc. plays an important role in the neutronics analysis. The possible sources of uncertainty in cross section are due to the variation of temperature, density, burn-up, porosity, energy group structure, resonance correction, etc. The measurement being not sufficient in the sense that reproducibility of the same value is questionable, uncertainty results in the measurement is categorized as epistemic. A fuzzy set theory based approach [Zadeh (1965)] can be used to quantify epistemic uncertainty [Ayyub and Gupta (1997)] that arises due to the imprecise variation of input parameters. The fuzzy set theory was introduced by Zadeh (1965)[Zadeh (1965)] with combination of classical set theory and proportional logic. Fuzzy logic [Ross (2010)], one of the soft computing methodology, uses partial membership function in contrast with binary membership function used in classical set theory. The major advantage of using partial membership function(i.e varies from 0 to 1.0) is to take into account the qualitative information e.g. hot, warm, cool, cold etc. to represent weather condition as understood by human brain and represented in natural language [Bernadette B. M. *et al.* (2010)]. There are various form of partial membership functions such as discrete and continuous. The membership function (MF) of a fuzzy variable represent its degree of impreciseness. In the present work, cross section data are considered as imprecise variable due to lack of knowledge of material property, temperature details, structural property, etc. The impact of imprecise definition of cross section data on neutron multiplication factors has been estimated by solving neutron diffusion equation numerically that conserve the neutron population in multiplying system by taking into account various processes involved in controlling chain reaction. Point neutron diffusion equation [Bera and Jagannathan (2007)], fuzzy set theory and results are discussed in the subsequent sections.

2. CALCULATION METHODOLOGY

2.1. *Point neutron diffusion equation for criticality calculation*

Neutron diffusion equation represents neutron balance between production and loss of neutron in a neutron multiplication system. Neutron diffusion equation, as mentioned in equation 1, contains leakage terms($D\nabla^2\phi$), absorption term ($\Sigma_a\phi$), production term ($\frac{\chi}{k_{eff}}\nu\Sigma_f\phi$) and scattering term ($\Sigma_s\phi$). In the equation 1, D is diffusion coefficient; Σ_f is fission cross section; Σ_a is absorption cross section, Σ_s is scattering cross section and ϕ is neutron flux. Combination of equation 1 and buckling equation 2 forms point neutron diffusion equation given in the equation 3. As the equation 3 does not contain any spatial variable, equation 3 is known as point neutron diffusion equation.

$$D\nabla^2\phi - \Sigma_a\phi + \frac{\chi}{k_{eff}}\nu\Sigma_f\phi + \Sigma_s\phi = 0 \tag{1}$$

$$\nabla^2\phi + B^2\phi = 0 \tag{2}$$

$$DB^2\phi + \Sigma_a\phi - \frac{\chi}{k_{eff}}\nu\Sigma_f\phi - \Sigma_s\phi = 0 \tag{3}$$

where, k is the neutron multiplication factor and characterizes the neutron multiplying system; χ is the fission spectrum and B is the geometrical buckling. The neutron energy spectrum in a neutron multiplication system varies from few 10^{-3} eV to the 10^{+6} eV. For the simplicity, five groups energy structure has been used for neutronic analysis. The cross section dataset, as shown in the following matrix, with five groups neutron energy structure contain (Σ_f), (Σ_a), transport cross section(Σ_{tr}), D and scattering cross section($\Sigma_{gg'}$). Each row containing 10 elements represents a particular neutron energy group denoted by the number 1 to 5. Column no 6 to 10 represents scattering cross section ($\Sigma_{gg'}$) with the energy group of incident neutron and scattered neutron are g and g' respectively.

$$\begin{bmatrix} \Sigma_{f1} & \Sigma_{a1} & \nu\Sigma_{f1} & \Sigma_{tr1} & D_1 & \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} & \Sigma_{15} \\ \Sigma_{f2} & \Sigma_{a2} & \nu\Sigma_{f2} & \Sigma_{tr2} & D_2 & \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} & \Sigma_{25} \\ \Sigma_{f3} & \Sigma_{a3} & \nu\Sigma_{f3} & \Sigma_{tr3} & D_3 & \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} & \Sigma_{35} \\ \Sigma_{f4} & \Sigma_{a4} & \nu\Sigma_{f4} & \Sigma_{tr4} & D_4 & \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44} & \Sigma_{45} \\ \Sigma_{f5} & \Sigma_{a5} & \nu\Sigma_{f5} & \Sigma_{tr5} & D_5 & \Sigma_{51} & \Sigma_{52} & \Sigma_{53} & \Sigma_{54} & \Sigma_{55} \end{bmatrix}$$

The representations of five group neutron diffusion equations are given in equation 5 to equation 9 with consideration of only down scattering cross section. Total fission neutron production is normalized to unity.

$$\sum_{g=1}^{5} \nu\Sigma_{fg}\phi = 1.0 \tag{4}$$

$$D_1 B^2 \phi_1 + \sigma_{R1}\phi_1 = \chi_1 \tag{5}$$

$$D_2 B^2 \phi_2 + \sigma_{R2}\phi_2 + \Sigma_{21}\phi_1 = \chi_2 \tag{6}$$

$$D_3 B^2 \phi_3 + \sigma_{R3}\phi_3 + \Sigma_{31}\phi_1 + \Sigma_{32}\phi_2 = \chi_3 \tag{7}$$

$$D_4 B^2 \phi_4 + \sigma_{R4}\phi_4 + \Sigma_{41}\phi_1 + \Sigma_{42}\phi_2 + \Sigma_{43}\phi_3 = \chi_4 \tag{8}$$

$$D_5 B^2 \phi_5 + \sigma_{R5}\phi_5 + \Sigma_{51}\phi_1 + \Sigma_{52}\phi_2 + \Sigma_{53}\phi_3 + \Sigma_{54}\phi_4 = \chi_5 \tag{9}$$

Linear equations in equation 5 to 9 are solved numerically to obtain energy group wise neutron flux by Gauss elimination method. The fluxes thus obtained are used to calculate neutron multiplication factor for infinite system ($k_{\propto}$) and effective neutron multiplication factor for finite system (k_{eff}) using equations 10 and 11 respectively.

$$K_{\propto} = \frac{\sum_{g=1}^{5} \nu\sigma_{fg}\phi_g}{\sum_{g=1}^{5} \sigma_{ag}\phi_g} \tag{10}$$

$$K_{eff} = \frac{\sum_{g=1}^{5} \nu\sigma_{fg}\phi_g}{\sum_{g=1}^{5} \sigma_{ag}\phi_g + \sum_{g=1}^{5} D_g B_g^2 \phi_g} \tag{11}$$

B_g^2 is the measured geometric buckling of a critical configuration which is derived from radial and axial dimensions and measured extrapolation distances in the two directions.

2.2. *Fuzzy Set Theory*

Each element of cross section dataset has been considered as imprecise variable. It has been represented by a fuzzy number set having triangular membership function with interval [l, m, r] as shown in the figure 1. where, l, m, r are left point, middle point and right point of triangular membership function. Mathematical formulation of this membership function is given in the equation 12

$$\mu_A(x) = \begin{cases} \frac{x-l}{m-l}, l \le x \le m \\ \frac{r-x}{r-m}, m \le x \le r \end{cases} \tag{12}$$

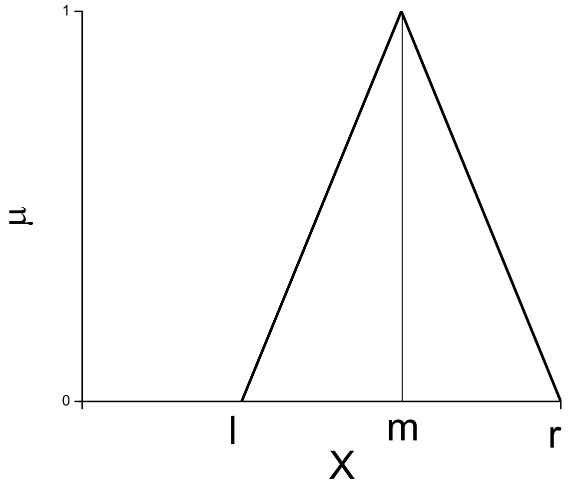

Fig. 1. Triangular Fuzzy membership function

Alpha-cut (α-cut) technique of a fuzzy set [Ross (2010)] provides the interval range corresponding to a specific value of membership function. Mathematically α-cut techniques is represented by

$$Cut_\alpha(A) = \{x|\mu_A(x) \geq \alpha\} \tag{13}$$

$$^\alpha X = [l + (m - l)\alpha, r - (r - m)\alpha] \tag{14}$$

In the uncertainty analysis, various nomenclatures are used to represent variation of a fuzzy parameter depending on the value of alpha. These are **core** ($\alpha =$ 1.0), **upper quartile**($\alpha = 0.75$), **cross over point** ($\alpha = 0.50$), **middle quartile** ($\alpha = 0.50$) and **lower quartile** ($\alpha = 0.25$).

2.3. *Overall Uncertainty Evaluation Methodology*

Each element of cross section dataset is considered as a fuzzy variables with triangular membership function. For a particular alpha-cut value, each element of cross section dataset will provide the lower bound and upper bound. Boundary values of cross section dataset will provide the uncertainty bound of neutron multiplication factor by solving point diffusion equation. An overall uncertainty methodology has been shown in the figure 2.

3. RESULTS AND DISCUSSIONS

Cross section dataset used in this analysis contain 10x5 elements. All fifty elements are considered as imprecise due to lack of knowledge of various parameters such

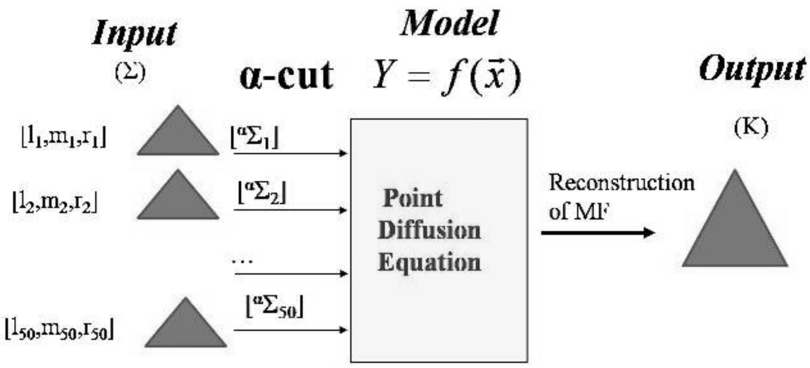

Fig. 2. Overall uncertainty evaluation methodology

as density, isotopic composition, temperature, resonance correction, etc. source of uncertainty of parameters such as cross section is due to the measurement with a finite degree of confidence. Measurement being not sufficient in the sense that reproducibility of the same value is questionable, uncertainty results in the measurement is categorized as epistemic. The degree of impreciseness of the each element is modeled with triangular membership function. The triangular membership function of fission cross section, which is the one element of cross section dataset, has been shown in the figure 3. In the figure 3, the bottom part is represented as the symmetric uncertainty of macroscopic fission cross section corresponding to the various α-cut values. The gray scale has been used to represent various degree of impreciseness. The degree of impreciseness decreases with increasing darkness.

The interval representation of the fission cross section of fast energy group is [0.1380159E-02, 0.1394100E-02, 0.1408041E-02]. For a particular α-cut value, two sets (i.e left and right boundary) of cross section dataset have been generated using equation 14. Each set of cross section data set have been used to estimate the neutronic properties of the neutron multiplying system. The output parameter, neutron multiplication factor becomes fuzzy variables due to the various mathematical operations on fuzzy variable. For the assessment of neutron multiplication factor, geometrical buckling is kept at a constant value equal to 0.0101 cm^{-2}. To obtain the complete variability of neutron multiplication factor, 10 α-cut values of all elements of cross section dataset are considered. The membership function of neutron multiplication factor has been generated through the accumulated k values computed with point neutron diffusion equation. The generated membership function of neutron multiplication factor is shown in the top graph of the figure 4. From the top graph, it is found that the membership function is asymmetrically distributed. To visualize the asymmetric uncertainty associated with it impreciseness, a grayscale contour has been formed and given in the bottom figure in the figure 4. In the gray scale contour, the increase of darkness represents the reduction of degree of

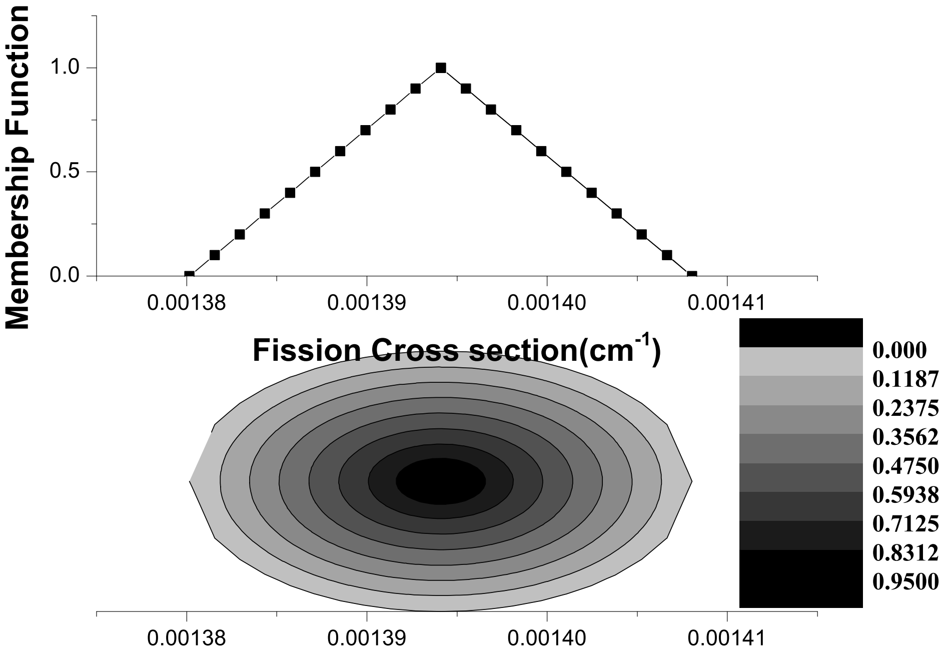

Fig. 3. Triangular MF of fission cross section and associated uncertainty

Table 1. Uncertainty bounds for neutron multiplication factor

	α-cut value	Lower bound	Upper bound
Crisp	1.0	1.1388880	1.1388880
Upper quartile	> 0.75	1.138861	1.138902
Middle quartile	> 0.50	1.1388340	1.1389150
Lower quartile	> 0.25	1.138863	1.138905
Support	> 0.0	1.1387800	1.1389430

impreciseness of the output variable e.g. neutron multiplication factor.

The asymmetric triangular membership function of neutron multiplication factor implies that for a particular α-cut value, the difference between crisp value and lower bound is larger than the difference between crisp value and upper bound. Various uncertainty bounds for various α-cut values are given in the 1.

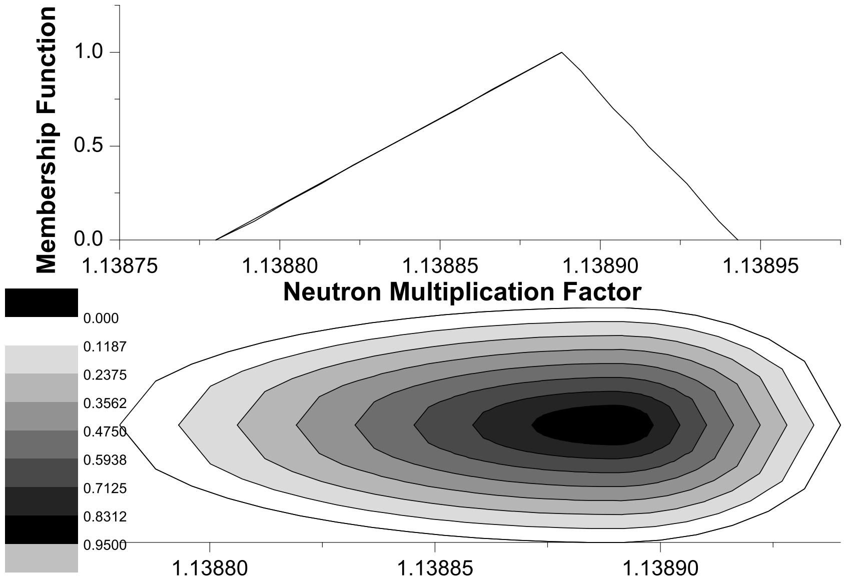

Fig. 4. MF of neutron multiplication factor and associated uncertainty

4. CONCLUSIONS

Fuzzy set theory based uncertainty analysis has been carried out for a neutron multiplying system containing fissile, fertile, moderator and reflector material. This methodology for uncertainty analysis captured epistemic uncertainty associated with imprecise nature of variation of various elements of neutron cross section dataset. With contrast to probabilistic uncertainty methodology, fuzzy set theory based uncertainty methodology captured the asymmetric variation of model parameter e.g. neutron multiplication factor. This asymmetric variability of neutron multiplication factor is arises due to the various mathematical operation with fuzzy variables. This innovative methodology can be applied to other engineering applications.

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