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Essential spectrum of the operators generated by PDE systems of stratified fluids and *Lp*-estimates for the solutions

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Abstract

We establish the localization and the structure of the spectrum of normal vibrations described by systems of partial differential equations modelling small displacements of stratified fluid in the homogeneous gravity field. We also compare the spectral properties of gravitational and rotational operators. The similarity of the essential spectrum for stratified and rotational flows corresponds to the analogy in the propagation of gravitational and Coriolis waves in viscous fluids, whose consideration includes the study of qualitative properties of the solutions, such as existence, uniqueness, smoothness, asymptotics, etc. We also obtain a solution of the Cauchy problem for a system of an exponentially stratified fluid in the gravity field in the form of singular integrals, taken in the Cauchy principal value sense, when singularities are removed by a ball, that is, isotropically. If the initial data have a specified smoothness, the solution is written in the form of integrals with weak singularities of the kernels. Both these forms of solutions enable exact L_p estimates $(p > 1)$ to be obtained.

Keywords*:* - Partial differential equations, essential spectrum, Sobolev spaces, stratified fluid, internal waves.

1. Introduction

Let us consider a PDE system which describes small displacements of an exponentially stratified viscous fluid in the gravity field

$$\begin{cases} r_* \dfrac{\partial \vec{u}}{\partial t} + \vec{e}_3 g r - m\Delta\vec{u} + \nabla p = 0 \\ \dfrac{\partial r}{\partial t} - \dfrac{N^2 r_*}{g} u_3 = 0 \\ div\,\vec{u} = 0 \end{cases} \qquad (1)$$

together with the PDE system describing the rotational movement of a viscous fluid

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + \vec{w}\times\vec{u} - m\Delta\vec{u} + \nabla p = 0 \\ div\,\vec{u} = 0 \end{cases} . \qquad (2)$$

Here $x \in \Omega \subset R^3$, $t \geq 0$, $\vec{u}(x,t) = (u_1, u_2, u_3)$ is the velocity field, *p(x,t)* is the scalar field of the dynamic pressure , $r(x,t)$ is the dynamic density, $\vec{w} = (0,0,w)$, $\vec{e}_3 = (0,0,1)$, and r_* , m , g , N , w are positive constants. The equations (1) are deduced under the assumption that the function of stationary distribution of density is performed by $r_* e^{-Nx_3}$. The system (2) describes the rotation over the vertical axis, $\vec{w}\times\vec{u}$ is the vector product in R^3 .

The systems (1) and (2) were studied from different angles , some of the results may be found in [2]- [5].

In [2] we prove that the essential spectrum of normal vibrations for the operators generated by (2) with $m = 0$, is the interval of the real axis $[-w, w]$, and we also construct an explicit example of non-uniqueness for the spectral parameter belonging to the essential spectrum.

In [3] the following result is stated:

Theorem 1.

The solution of a Cauchy problem for (2) has the following asymptotic property : the velocity field decreases as $\left(1/t\right)^{5/2}$, $t \to \infty$, where the decay of order $\left(1/t\right)^{3/2}$ is due to the viscosity and the influence of the Coriolis term is $1/t$.

In [4], [5], [7] we prove that for the system (1) the distribution of energy is the same. Namely, from the point of view of *t*-asymptotics, the effects of gravitation and rotation are analogous in viscous fluids:

Theorem 2.

Let us consider the system (1) in the semi-space

$$R_+^3 = \{(x_1, x_2, x_3) : (x_1, x_2) \in R^2, x_3 \geq 0\},$$

together with the boundary conditions

$$\left.\frac{\partial u_1}{\partial x_3}\right|_{x_3=0} = \left.\frac{\partial u_2}{\partial x_3}\right|_{x_3=0} = u_3|_{x_3=0} = 0 . \qquad (3)$$

Then, for certain initial conditions, the solution of (1),(3) has the following asymptotic representation :

$$\vec{u}(x,t)=\frac{\vec{C}(x)}{(m\,t)^{\frac{3}{2}}(Nt)}+o\left(t^{-\frac{5}{2}}\right),\quad t\to\infty\quad .$$

Let us observe that the mentioned analogy between gravitational and rotational waves in the dissipation of energy, leads to the corresponding analogy in spectral properties. Indeed, for the systems (1) and (2) with $m=0$, the singular solutions have the following forms, respectively :

$$E(x,t)=\frac{1}{4p|x_3|}\int_0^{\frac{N\,t|x_3|}{|x|}} J_0(a)da \quad , \qquad \text{and}$$

$$E(x,t)=\frac{1}{4p|\bar{x}|}\int_0^{\frac{w\,t|\bar{x}|}{|x|}} J_0(a)da \quad , |\bar{x}|=\sqrt{x_1^2+x_2^2}\ .$$

Summing up all these results, it seems appropriate to express the conjecture that the operators generated by the system (1) should possess spectral properties, analogous to the system (2) , namely, the essential spectrum of such operators should be the interval [-*N*,*N*] . In this paper we prove that this conjecture is true.

For system (1) we will construct the explicit form of the solution of the Cauchy problem in form of singular integrals which will allow us to obtain L_p -estimates by means of the Calderón-Zygmund Theorem.

2. Spectral Problem Formulation

Let us consider the system

$$\begin{cases} r_*\dfrac{\partial\vec{u}}{\partial t}+\vec{e}_3 gr+\nabla p=0 \\ \dfrac{\partial r}{\partial t}-\dfrac{N^2 r_*}{g}u_3=0 \\ div\,\vec{u}=0 \end{cases} \quad . \qquad (4)$$

Differentiating the second equation of (4) with respect to t , we obtain

$$\begin{cases} \dfrac{\partial^2\vec{u}}{\partial t^2}+\vec{e}_3 N^2 u_3+\nabla P=0 \\ div\,\vec{u}=0 \end{cases} \quad , \qquad (5)$$

where $P=\dfrac{1}{r_*}\dfrac{\partial p}{\partial t}$. For the system (4), let us consider the boundary value problem

$$\vec{u}\cdot\vec{n}\big|_{\partial\Omega}=0 \quad , \qquad (6)$$

where $\vec{n}$ is the vector of the external normal for the bounded domain $\Omega\subset R^3$.

Let $G(\Omega)$ be the space of potential fields in $L_2(\Omega)$:
$G_2(\Omega) = \{\vec{u} \in L_2(\Omega) : \vec{u} = \nabla j \, ; j \in W_2^1(\Omega)\}$.

Furthermore, let $\overset{0}{J}(\Omega)$ be the space of solenoidal fields :
$\overset{0}{J}(\Omega) = \{\vec{u} \in C^1(\Omega) : div\, \vec{u} = 0 \, , \vec{u} \cdot \vec{n}\big|_{\partial\Omega} = 0\}$.

Finally, let us introduce the space $J_2(\Omega)$ as a closure of $\overset{0}{J}(\Omega)$ in the norm of $L_2(\Omega)$.
It can be shown ([1]), that $L_2(\Omega)$ permits the following orthogonal decomposition:

$$L_2(\Omega) = J_2(\Omega) \oplus G_2(\Omega) \; .$$

Let P be the operator of the orthogonal projection of $L_2(\Omega)$ onto $J_2(\Omega)$. Now, let us define the operator B :

$$B\vec{u} = P\{u_3 \vec{e}_3\}$$

with the domain $D(B) = J_2(\Omega)$.
Thus, the system (5) transforms into

$$\begin{cases} \dfrac{\partial^2 \vec{u}}{\partial t^2} + N^2 B\vec{u} = 0 \\ \vec{u} \in J_2(\Omega) \end{cases} . \qquad (7)$$

For the system (7) we consider the problem of normal vibrations

$$\vec{u}(x,t) = \vec{v}(x) e^{ilt} \quad . \qquad (8)$$

Therefore, we can finally write the system (7) as

$$\begin{cases} l^{\;2} \vec{v} - N^2 B\vec{v} = 0 \\ \vec{v} \in J_2(\Omega) \end{cases} . \qquad (9)$$

Our aim is to investigate the spectrum of the operator B . From the physical point of view, the separation of variables (8) serves as a tool to establish the possibility to represent every non-stationery process described by (4) as a linear superposition of the normal vibrations. The knowledge of the spectrum of the normal vibrations, its structure and localization, may be very useful for studying the stability of the flows. Finally, the spectrum of operator B is important in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operator B .

3. Spectral Problem Solution

Lemma 3. B is a positive self-adjoint operator in $J_2(\Omega)$.
Proof. Evidently, $\|B\vec{u}\|_{L_2(\Omega)} \le \|\vec{u}\|_{L_2(\Omega)}$ and thus $\|B\| \le 1$. Let $\vec{u}\,, \vec{v} \in J_2(\Omega)$. Then,

$$(\vec{u}, B\vec{v}) = (\vec{u}, P\{v_3\vec{e}_3\}) = (P\vec{u}, \{v_3\vec{e}_3\}) = \int_\Omega u_3 v_3 dx = (B\vec{u}, \vec{v}).$$

Since B is bounded, its self-adjointness follows from its symmetry. Finally,
$(\vec{u}, B\vec{u}) = \int_\Omega |u_3(x)|^2 dx \ge 0$,

which concludes the proof.

Lemma 4. The kernel of B is the subspace $H_J(\Omega)$ which consists of all elements of $J_2(\Omega)$ with trivial third component.

Proof. Obviously, $H_J(\Omega) \subset Ker(B)$.

Suppose that $\vec{u} \in Ker(B)$ and $\vec{u} \notin H_J(\Omega)$. Then, we obtain that $(\vec{u}, B\vec{u}) = \int_\Omega |u_3(x)|^2 dx = 0$, which implies $u_3 = 0$ and thus $H_J(\Omega) = Ker(B)$.

Corollary. $l = 0$ is an eigenvalue of infinite multiplicity for B. Its corresponding eigenvectors compose all the subspace $H_J(\Omega)$.

Now, let us consider the same separation of variables for the function $P(x,t)$:

$$P(x,t) = q(x)e^{ilt}, \quad q \in W_2^1(\Omega).$$

If $q(x)$ is a solution of the system

$$\begin{cases} -l^2 v_1 + \dfrac{\partial q}{\partial x_1} = 0 \\ -l^2 v_2 + \dfrac{\partial q}{\partial x_2} = 0 \\ \left(-l^2 + N^2\right)v_3 + \dfrac{\partial q}{\partial x_3} = 0 \\ \dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0 \end{cases}, \tag{10}$$

then $q(x)$ satisfies the equation $\Delta q = -div\left(N^2 v_3 \vec{e}_3\right)$, which implies

$$div\left(N^2 v_3 \vec{e}_3 + \nabla q\right) = 0.$$

Thus, the projection operator B obtains its explicit form as $N^2 B\vec{v} = N^2 v_3 \vec{e}_3 + \nabla q$.

We shall establish now the structure of the spectrum of the operator B.

Theorem 5. The essential spectrum of the operator $N^2 B$ is the interval of the real axis $[-N, N]$. Moreover, the points $0, \pm N$ are eigenvalues of infinite multiplicity.

Proof. First we recall that the essential spectrum is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity ([8]). We shall use the following criterion which is attributed to Weyl ([8]): A necessary and sufficient condition that a real finite value m be a point of the essential spectrum of a self-adjoint operator B is that there exist a sequence of elements $x_n \in D(B)$ such that

$$\begin{aligned} &\|x_n\| = 1, \; x_n \to 0 \text{ weakly and} \\ &\|(B - mI)x_n\| \to 0 \end{aligned}. \tag{11}$$

Let us denote $l^2 = m$, $m \neq 0$. Then, the system (10) takes the matrix form

$$\begin{pmatrix} -m & 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & -m & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & N^2-m & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} . \qquad (12)$$

One can easily see that the main symbol of the differential operator in (12) is

$$L(x) = \begin{pmatrix} -m & 0 & 0 & x_1 \\ 0 & -m & 0 & x_2 \\ 0 & 0 & N^2-m & x_2 \\ x_1 & x_2 & x_3 & 0 \end{pmatrix} .$$

As $\det L(x) = m\left(-m|x|^2 + N^2|\bar{x}|^2\right)$, $|\bar{x}|^2 = x_1^2 + x_2^2$, $|x|^2 = x_1^2 + x_2^2 + x_3^2$,

we may conclude that the operator $N^2 B$ is not elliptic in sense of Douglis -Nirenberg if and only if $m \in \left[0, N^2\right]$, see ([6]).

Now, let us consider $m_0 \in \left(0, N^2\right)$ and choose a vector x such that

$$-m_0|x|^2 + N^2|\bar{x}|^2 = 0 \ , \ |x| \neq 0 \ .$$

Therefore, there exists $h = (h_1, h_2, h_3, h_4)$, $h_i \neq 0$, $1 \le i \le 4$; such that $L(x)h = 0$:

$$\begin{cases} -m_0 h_1 + x_1 h_4 = 0 \\ -m_0 h_2 + x_2 h_4 = 0 \\ \left(-m_0 + N^2\right) h_3 + x_{31} h_4 = 0 \\ x_1 h_1 + x_2 h_2 + x_3 h_3 = 0 \end{cases} \qquad (13)$$

Solving (13) with respect to h , we obtain

$$\begin{cases} h_1 = \dfrac{x_1}{m_0} \ , \ h_2 = \dfrac{x_2}{m_0} \\ h_3 = \dfrac{x_3}{m_0 - N^2} \ , \ h_4 = 1 \end{cases} \qquad (14)$$

We observe that $h_i \neq 0$, $1 \le i \le 4$. Now, let us choose a function

$$y_0(x) \in C_0^\infty(\Omega) \ , \ \int_{|x| \le 1} y_0^2(x)dx = 1 \ .$$

We fix $x_0 \in \Omega$ and define $y_k(x) = k^{\frac{3}{2}} y_0(k(x - x_0))$, $k = 1,2,...$

One can easily see that

$$\|y_k\|_{L_2(\Omega)} = 1 \ , \ \left\|\frac{\partial y_k}{\partial x_j}\right\|_{L_2(\Omega)} = C_j^1 k \ , \left\|\frac{\partial^2 y_k}{\partial x^2{}_j}\right\|_{L_2(\Omega)} = C_j^2 k^2 \ , \quad (15)$$

where the constants $C_j^i \neq 0$ do not depend on k. We define the Weyl sequence $\tilde{v}^k = \left(v_1^k, v_2^k, v_3^k, q^k\right)$
as follows :

$$\begin{cases} v_j^k(x) = h_j e^{ik^3<x,x>}\left(y_k - \dfrac{1}{ik^3 x_j}\dfrac{\partial y_k}{\partial x_j}\right), \ j = 1,2,3 \\ q^k(x) = -\dfrac{i}{k^3} y_k \, e^{ik^3<x,x>} \\ <x,x> = x_1x_1 + x_2x_2 + x_3x_3 \ , \ k = 1,2,.... \end{cases} \quad (16)$$

Now we have to verify that the sequence $\tilde{v}^k$ defined above, satisfies the conditions (11). Note that a Weyl sequence is an explicit solution of a system of partial differential equations.
For the functions (16), the weak convergence to zero is evident. Let us introduce the matrix differential operator M :

$$M = \begin{pmatrix} 0 & 0 & 0 & \frac{\partial}{\partial x_1} \\ 0 & 0 & 0 & \frac{\partial}{\partial x_2} \\ 0 & 0 & N^2 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \end{pmatrix}.$$

Thus, the system (12) can be expressed as $(M - mI_3)\tilde{v} = 0$, where

$$I_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let us prove that $\lim\limits_{k\to\infty}\left\|\tilde{f}^k\right\|_{L_2(\Omega)} = 0$, where $\tilde{f}^k = (M - m_0 I_3)\tilde{v}^k$.

We have

$$f_1^k = -m_0 v_1^k + \frac{\partial q^k}{\partial x_1} =$$

$$= \left(-m_0 h_1 + x_1 h_4\right) \times y_k\, e^{ik^3 <x,x>} - \frac{i}{k^3} e^{ik^3 <x,x>} \frac{\partial y_k}{\partial x_1} \left[\frac{m_0 h_1}{x_1} + 1\right].$$

From (13),(14) we have $-m_0 h_1 + x_1 h_4 = 0$. Therefore,

$$\left\| f_1^2 \right\|_{L_2(\Omega)} \le Const \cdot k^{-2} \to 0 \text{ as } k \to \infty \; .$$

Analogously,

$$f_2^k = -m_0 v_2^k + \frac{\partial q^k}{\partial x_2} =$$

$$= \left(-m_0 h_2 + x_2 h_4\right) \times y_k\, e^{ik^3 <x,x>} - \frac{i}{k^3} e^{ik^3 <x,x>} \frac{\partial y_k}{\partial x_2} \left[\frac{m_0 h_2}{x_2} + 1\right].$$

From (13),(14) it follows that $-m_0 h_2 + x_2 h_4 = 0$. Thus,

$$\left\| f_2^2 \right\|_{L_2(\Omega)} \le Const \cdot k^{-2} \to 0 \text{ as } k \to \infty \; .$$

In the similar way, $\left(-m_0 + N^2\right) h_3 + x_3 h_4 = 0$ implies

$$\left\| f_3^2 \right\|_{L_2(\Omega)} \le Const \cdot k^{-2} \to 0 \text{ as } k \to \infty \; .$$

Analogously, from $< x, h >= 0$ we obtain $\lim\limits_{k\to\infty} \left\| f_4^2 \right\|_{L_2(\Omega)} = 0$.

To verify that the norms $\left\| \tilde{v}^k \right\|_{L_2(\Omega)}$ are separated from zero, it is sufficient to prove that at least the norms of one component of the field $\tilde{v}^k$ are separated from zero as $k \to \infty$.

Let us consider the two summands $v_1^k = v_{11}^k + v_{12}^k$,

where $v_{11}^k = h_1 e^{ik<x,x>} y_k(x)$, $v_{12}^k = h_1 e^{ik<x,x>} \dfrac{i}{k^3 x_1} \dfrac{\partial y_k(x)}{\partial x_1}$.

Evidently, $\lim\limits_{k\to\infty} \left\| v_{12}^k \right\|_{L_2(\Omega)} = \lim\limits_{k\to\infty} \dfrac{|h_1|}{k^3 |x_1|} \left\| \dfrac{\partial y_k}{\partial x_1} \right\|_{L_2(\Omega)} = 0$.

However, $\left\| v_{11}^k \right\|_{L_2(\Omega)} = \left\| h_1 y_k\, e^{ik^3 <x,x>} \right\|_{L_2(\Omega)} = |h_1| \left\| y_k \right\|_{L_2(\Omega)} = |h_1| \ne 0$.

In this way, we have proved that the sequence (16) satisfies Weyl conditions (11). Since the essential spectrum is closed, the points $m = 0$, N^2, belong to it. Returning to the initial spectral parameter l , we obtain that the essential spectrum of the operator $N^2 B$ is the interval $[-N, N]$.

We have seen that $l = 0$ is an eigenvalue of infinite multiplicity . The same statement holds for the points $l = \pm N$.

Indeed, for $l = \pm N$ the system (10) transforms into

$$
\begin{cases}
-N^2 v_1 + \partial q / \partial x_1 = 0 \\
-N^2 v_2 + \partial q / \partial x_2 = 0 \\
\partial q / \partial x_3 = 0 \\
\partial v_1 / \partial x_1 + \partial v_2 / \partial x_2 + \partial v_3 / \partial x_3 = 0
\end{cases}
.
$$

It can be easily seen that any function of the type $(0,0, j(x_1, x_2),0)$, $j \in C_0^\infty$, satisfies the last system.
Thus, theorem 5 is proved.

4. Construction of solutions for the Cauchy problem

We consider a system of equations of the form

$$
\begin{cases}
r_* \dfrac{\partial v_1}{\partial t} + \dfrac{\partial p}{\partial x_1} = 0 \\
r_* \dfrac{\partial v_2}{\partial t} + \dfrac{\partial p}{\partial x_2} = 0 \\
r_* \dfrac{\partial v_3}{\partial t} + gr + \dfrac{\partial p}{\partial x_3} = 0 \\
\dfrac{\partial r}{\partial t} - \dfrac{N^2 r_*}{g} v_3 = 0 \\
\dfrac{\partial v_1}{\partial x_1} + \dfrac{\partial v_2}{\partial x_2} + \dfrac{\partial v_3}{\partial x_3} = 0
\end{cases}
\qquad (17)
$$

in the domain $\{x \in R^3, t > 0\}$, where $\vec{v}(x,t)$ is a velocity field with components $v_1(x,t), v_2(x,t), v_3(x,t)$, $p(x,t)$ is the scalar field of the dynamic pressure, $r(x,t)$ is the dynamic density and $r_*\ g, N$ are positive constants .

Let us consider first the Cauchy problem for (17) :

$$
\begin{aligned}
&\vec{v}\big|_{t=0} = \vec{v}^{\,0}(x) \\
&r\big|_{t=0} = 0
\end{aligned}
\quad . \qquad (18)
$$

Differentiating the fourth equation of (17) with respect to t , we obtain

$$
\begin{cases}
\dfrac{\partial^2 \vec{v}}{\partial t^2} + \nabla P + \vec{e}_3 N^2 v_3 = 0 \\
div(\vec{v}) = 0
\end{cases}
, \qquad (19)
$$

where $\vec{e}_3 = (0,0,1)$, $P = \dfrac{1}{r_*} \dfrac{\partial p}{\partial t}$.

If we denote by P the operator of the orthogonal projection of $L_2(R^3)$ onto $J_2(R^3)$, then we can define the following operator A :

$$A\vec{v} = P\{v_3\vec{e}_3\} \qquad (20)$$

with the domain $D(A) = J_2(R^3)$.

Since $\|A\vec{v}\| \le \|\vec{v}\|$, the norm of the operator A is not greater than unity. Thus, the equation

$$\frac{d^2\vec{v}}{dt^2} + A\vec{v} = 0 \ , \ \vec{v}\big|_{t=0} = \vec{v}^{\,0} \ , \ \frac{d\vec{v}}{dt}\bigg|_{t=0} = \vec{v}_1^{\,0} \qquad (21)$$

has the solution

$$\vec{v} = \vec{v}^{\,0}\cos ANt + \frac{\vec{v}_1^{\,0}}{N}\sin ANt =$$

$$= \vec{v}^{\,0} - \frac{N^2t^2}{2!}A^2\vec{v}^{\,0} + \frac{N^4t^4}{4!}A^4\vec{v}^{\,0} - .. \qquad (22)$$

$$.. + \frac{1}{N}\left[\frac{Nt}{1!}A\vec{v}_1^{\,0} - \frac{N^3t^3}{3!}A^3\vec{v}_1^{\,0} + \frac{N^5t^5}{5!}A^5\vec{v}_1^{\,0} - ..\right].$$

The two series in (22) converge uniformly with respect to t , since the n-th term satisfies the inequalities $\|A^n\vec{v}_0\| \le \|\vec{v}_0\|$, $\left\|\frac{N^nt^n}{n!}A^n\vec{v}^{\,0}\right\| \le \frac{(Nt)^n}{n!}\|\vec{v}^{\,0}\|$ either for $\vec{v}^{\,0}$, or for $\vec{v}_1^{\,0}$.

Obviously, the series of the second derivatives of (22) converges uniformly with respect to t and the initial conditions (21) are satisfied. In this way, the series (22) represent the solution of the Cauchy problem. It is easy to see that the problem is well-posed. In fact, let us verify the continuous dependence of the solution with respect to the initial data.

Let $$\|\vec{v}^{\,0} - \vec{v}_*^{\,0}\| < e \ , \ \|\vec{v}_1^{\,0} - \vec{v}_{1*}^{\,0}\| < e \ .$$

For the vectors $\vec{v} = \vec{v}^{\,0}\cos ANt + \frac{\vec{v}_1^{\,0}}{N}\sin ANt$ and $\vec{v}_* = \vec{v}_*^{\,0}\cos ANt + \frac{\vec{v}_{1*}^{\,0}}{N}\sin ANt$ we will have

$\|\vec{v} - \vec{v}_*\| \le \max\left(1, \frac{1}{N}\right)\left(\|\cos ANt(\vec{v}^{\,0} - \vec{v}_*^{\,0})\| + \|\sin ANt(\vec{v}_1^{\,0} - \vec{v}_{1*}^{\,0})\|\right) < Ce^{Nt}e$. Thus, the Cauchy problem is well-posed.

Now we shall construct the explicit form of the solution of the Cauchy problem for (17).

For system (17) we consider the initial conditions (18) and the additional conditions of the absence of the rotational component in (x_1, x_2)

$$\frac{\partial v_1^0}{\partial x_2} - \frac{\partial v_2^0}{\partial x_1} = 0 \ , \qquad (23)$$

together with the natural condition

$$div(\vec{v}^{\,0}) = 0 \ . \qquad (24)$$

So that we shall be dealing with convergent integrals, we assume that the initial data have, for example, continuous second derivatives and decrease sufficiently rapidly at infinity together with their derivatives up to the second order. Using the Fourier transform with respect to x , the Laplace transform with respect to t and the conditions (23), (24), we obtain the solution of our problem in the form

$$\hat{\hat{v}}(x,l)=\frac{l|x|^2}{l^2|x|^2+|x'|^2}\hat{\hat{v}}^0(x)\ ,\quad \hat{r}(x,l)=\frac{|x|^2\hat{v}_3^0(x)}{g\left(l^2|x|^2+|x'|^2\right)}\ ,\quad \hat{p}(x,l)=\frac{ix_3\hat{v}_3^0(x)}{l^2|x|^2+|x'|^2}\ ,$$

where $x=(x_1,x_2,x_3)$, $|x|^2=\sum_{k=1}^{3}x_k^2$, $|x'|^2=\sum_{k=1}^{2}x_k^2$. After an inverse Laplace transform we obtain the solution in the form

$$\hat{\hat{v}}(x,t)=\hat{\hat{v}}^0(x)\cos\frac{|x'|}{|x|}t,\ \hat{r}(x,t)=\hat{v}_3^0(x)\frac{|x|}{g|x'|}\sin\frac{|x'|}{|x|}t,\ \hat{P}(x,t)=\hat{v}_3^0(x)\frac{ix_3}{|x'||x|}\sin\frac{|x'|}{|x|}t\ . \quad (25)$$

We now find the inverse Fourier transform of the required solution. We first obtain the solution in the form of integrals with weak singularities of the kernels. For this we seek a vector $\vec{v}(x,t)$ expressed in terms of the Laplace operator and a function $p(x,t)$ expressed in terms of the first derivatives of the initial data. As we see from (25), it is sufficient to calculate only two kernels

$$K_1(x-y,t)=\frac{1}{(2p)^3}\int_{-\infty}^{\infty}e^{i(x,x-y)}\frac{1}{|x|^2}\cos\frac{|x'|}{|x|}t\,dx\ , \qquad (26)$$

$$K_2(x-y,t)=\frac{1}{(2p)^3}\int_{-\infty}^{\infty}e^{i(x,x-y)}\frac{1}{|x'||x|}\sin\frac{|x'|}{|x|}t\,dx\ . \qquad (27)$$

We note that K_2 is the primitive of K_1 with respect to t , and it is therefore sufficient to calculate only one of these integrals.

An integral of type (27) is calculated in [11] by means of Sonine´s formulas for Bessel functions and is given by

$$K_2(x-y,t)=\frac{1}{4p}\frac{1}{r}\int_0^t J_0(t-t)J_0\left(\frac{r\,t}{r}\right)dt\ , \qquad (28)$$

where $r^2=(x_3-y_3)^2$, $r^2=\sum_{k=1}^{3}(x_k-y_k)^2$, and J_0 is the Bessel function of order zero. Therefore

$$K_1(x-y,t)=\frac{1}{4p}\frac{1}{r}J_0\left(\frac{r\,t}{r}\right)-\frac{1}{4p}\frac{1}{r}\int_0^t J_1(t-t)J_0\left(\frac{r\,t}{r}\right)dt\ . \qquad (29)$$

If we now use (28) and (29), the solution of the Cauchy problem for the system (3) can be written in the form

$$\vec{v}(x,t)=\iiint_{R^3}\left\{-\Delta\vec{v}^0(y)K_1(x-y,t)\right\}dy\ , \qquad (30)$$

$$P(x,t)=\iiint_{R^3}\left\{\frac{\partial v_3^0}{\partial y_3}K_2(x-y,t)\right\}dy\ . \qquad (31)$$

These are basic formulas defining our solution with weak singularities of the kernels. To obtain the limit exact estimates in L_p -norms for the solution of the Cauchy problem it is helpful to rewrite these formulas in another form with strong singularities of the kernels.

In (30) and (31) we integrate by parts in order that the solution should be expressed in terms of the initial functions rather than their derivatives. This is easily done in formula (31) for $p(x,t)$, because after one integration by parts the kernels will still have an integrable singularity. In (30), however, after the second integration by parts we shall have a strong (locally non-integrable) singularity.

We remove from our space the ball K_e of radius e with boundary S_e and center at (x_1, x_2, x_3) and denote the rest of the domain by Ω_e. The component $v_1(x,t)$ of $\vec{v}(x,t)$ is given by

$$v_1(x,t) = \lim_{e\to 0}\frac{1}{4p}\iiint_{\Omega_e}\left(-\Delta v_1^0\right)\left[\frac{1}{r}J_0\left(\frac{rt}{r}\right) - \frac{1}{r}\int_0^t J_1(t-t)J_0\left(\frac{rt}{r}\right)dt\right]dy =$$

$$= \lim_{e\to 0}\frac{1}{4p}\iiint_{\Omega_e}\left(-v_1^0\right)\left[\Delta\left(\frac{1}{r}J_0\left(\frac{rt}{r}\right)\right) - \int_0^t J_1(t-t)\Delta\left(\frac{1}{r}J_0\left(\frac{rt}{r}\right)\right)dt\right]dy +$$

$$+ \lim_{e\to 0}\frac{1}{4p}\iint_{S_e}\left(-v_1^0\right)\left[\frac{\partial}{\partial n}\left(\frac{1}{r}J_0\left(\frac{rt}{r}\right)\right) - \int_0^t J_1(t-t)\frac{\partial}{\partial n}\left(\frac{1}{r}J_0\left(\frac{rt}{r}\right)\right)dt\right]ds,$$

where *n* is the normal to the surface S_e which is interior with respect to Ω_e.

We calculate the principal value of the integral over the surface of the sphere and denote it by I. We note that on the surface of the sphere

$$-\frac{\partial}{\partial n}\left(\frac{1}{r}J_0\left(\frac{rt}{r}\right)\right) = \frac{1}{r^2}J_0\left(\frac{rt}{r}\right).$$

We transform to spherical coordinates on the surface of the sphere of radius e :

$$\begin{aligned} y_1 - x_1 &= e\cos j\sin q \\ y_2 - x_2 &= e\sin j\sin q \\ y_3 - x_3 &= e\cos q \\ ds &= e^2\sin q\,dq\,dj \end{aligned},$$

so that

$$I = \frac{1}{4p}\lim_{e\to 0} v_1^0(x)\iint_{S_e}\left[\frac{1}{r^2}J_0\left(\frac{rt}{r}\right) - \left(\int_0^t J_1(t-t)\frac{1}{r^2}J_0\left(\frac{rt}{r}\right)dt\right)\right]ds =$$

$$v_1^0(x)\int_0^{p/2}\left[J_0(t\cos q)\sin q - \left(\int_0^t J_1(t-t)J_0(t\cos q)dt\right)\sin q\right]dq = v_1^0(x)\Phi(t),$$

where

$$\Phi(t) = \frac{1}{t}\int_0^t J_0(h)dh - \int_0^t J_1(t-t)\left(\frac{1}{t}\int_0^t J_0(h)dh\right)dt. \qquad (32)$$

On carrying out exactly similar arguments for the other components of $\vec{v}(x,t)$ and $P(x,t)$, we obtain formulas for the solution of the Cauchy problem for system (19) in the form

$$\vec{v}(x,t) = \vec{v}^{\,0}(x)\Phi(t) +$$
$$+ \frac{1}{4p} Vp \iiint \vec{v}^{\,0}(y) \left(\Delta\left[-\frac{1}{r} J_0\left(\frac{rt}{r}\right)\right] + \int_0^t J_1(t-t)\Delta\left(\frac{1}{r} J_0\left(\frac{r\,t}{r}\right)\right) dt \right) dy \qquad (33)$$

where the integrals are taken in the principal value sense over the sphere, and the function $\Phi(t)$ is given by (32).

For $P(x,t)$ we obtain

$$P(x,t) = \frac{1}{4p} \iiint v_3^0(y) \frac{\partial}{\partial y_3} \left\{ -\frac{1}{r} \int_0^t J_0(t-t) J_0\left(\frac{r\,t}{r}\right) dt \right\} dy \quad , \qquad (34)$$

where the kernels have integrable singularities.

In what follows we shall show that the function $\vec{v}(x,t)$ defined by (33) is a unique solution of the Cauchy problem and that $p(x,t)$ is defined by (34) to within a term depending on t (since the initial Cauchy data for it were not given).

5. L*p*- estimates for a solution of the Cauchy problem

To obtain L_p -estimates for a solution of the Cauchy problem we shall show that kernels which are used in writing out the solution and its derivatives satisfy the conditions of the Calderón-Zygmund Theorem. We write (33) in the form of convolution:

$$\vec{v}(x,t) = \vec{v}^{\,0}(x)\Phi(t) + \left(\vec{v}^{\,0} * \Gamma\right)_{R^3} \quad , \qquad (35)$$

where

$$\left(\vec{v}^{\,0} * \Gamma\right)_{R^3} = \iiint_{R^3} \vec{v}^{\,0}(y)\Gamma(x-y,t)dy \quad , \qquad (36)$$

$$\Gamma(x,t) = G(x,t) - \int_0^t J_1(t-t) G(x,t)dt \quad , \qquad (37)$$

where the infinite triple integrals are taken in the sense of principal value, and from (33), after the corresponding differentiation, we obtain

$$G(x,t) = \frac{1}{4p}\left[\frac{t^2\left(x_1^2 + x_2^2\right)}{r^5} J_0\left(\frac{rt}{r}\right) + \frac{t}{r^3}\left(\frac{r}{r} + \frac{r}{r}\right) J_0'\left(\frac{rt}{r}\right)\right] \quad . \qquad (38)$$

It is easy to see that G is infinitely differentiable function of t (a singularity in the space $x = (x_1, x_2, x_3)$ does not increase on differentiation with respect to t).

We examine the properties of the kernel G for any finite t : $0 \le t \le T < \infty$.

1) G is a homogeneous function of x of degree -3 . The proof is obvious on nothing that Bessel functions of the argument rt/r are homogeneous functions of degree zero.

2) G may be put in the form $\Gamma(x,t)=\dfrac{\tilde{\Omega}(x,t)}{r^3}$, where

$$\tilde{\Omega}(x,t)=\Omega(x,t)-\int_0^t J_1(t-t)\Omega(x,t)dt \quad ,$$

$$\Omega(x,t)=\frac{1}{4p}\left[\frac{t^2\left(x_1^2+x_2^2\right)}{r^2}J_0\left(\frac{rt}{r}\right)+\frac{t\left(r^2+r^2\right)}{rr}J_0'\left(\frac{rt}{r}\right)\right].$$

3) The integrals of $\tilde{\Omega}(x,t)$ over the unit sphere are zero.

For, transforming to polar coordinates on the unit sphere and using the change of variables $\cos q = z$, we obtain

$$\iint_{r=1}\Omega ds=\frac{1}{4p}\iint_{r=1}\left\{t^2\left(x_1^2+x_2^2\right)J_0(rt)+t\left[r+\frac{1}{r}\right]J_0'(rt)\right\}ds=$$

$$=\int_0^{p/2}\left\{t^2\sin^3 q\ J_0(t\cos q)+t\sin q\left[\cos q+\frac{1}{\cos q}\right]J_0{}'(t\cos q)\right\}dq=$$

$$=\int_0^1\left\{t^2\left(1-z^2\right)J_0(tz)+t\left[z+\frac{1}{z}\right]J_0{}'(tz)\right\}dz=$$

$$=\int_0^1 t^2\left(1-z^2\right)J_0(tz)dz-\int_0^1 t^2\left(1+z^2\right)\left(J_0(tz)+J_0{}''(tz)\right)dz=$$

$$=\int_0^1 J_0(tz)\left(-2t^2z^2\right)dz-\int_0^1 t^2\left(1+z^2\right)J_0{}''(tz)dz\ . \qquad (39)$$

In (26) we used the Bessel equation $J_0''(y)+\dfrac{J_0'(y)}{y}+J_0(y)=0$. Integrating by parts and using the Bessel equation for the first integral in (39), we obtain

$$-\int_0^1 2t^2z^2J_0(tz)dz=2\int_0^1 t^2z^2J_0''(tz)dz+2\int_0^1 tzJ_0'(tz)dz=2tJ_0'(t)-2\int_0^1 tzJ_0'(tz)dz\ .$$

For the second integral in (39), we integrate by parts and obtain

$$\int_0^1 t^2\left(1+z^2\right)J_0{}''(tz)dz=2tJ_0'(t)-2\int_0^1 tzJ_0'(tz)dz\ .$$

Finally, summing up the two last results, we have $\iint_{r=1}\Omega ds=0$.

Thus the conditions of the Calderón-Zygmund Theorem [12] are satisfied, and we therefore have the following estimate for the vector $\vec{v}(x,t)$ in the L_p -norm, $1<p<\infty$, in the layer $E_4^T=\{-\infty<x_i<+\infty,\ 0\le t\le T\}$ (and also on each cross-section t =const):

$$\|\vec{v}\|_{L_p(E_4^T)} \leq C(p,T)\|\vec{v}^{\,0}\|_{L_p(R^3)} \quad , \qquad (40)$$

where C depends only on p and T . We observe that, in differentiation with respect to t , the properties 1), 2) and 3) for the kernel G in (35)-(38) are preserved. In this way, for any derivative with respect to t we shall have

$$\|D_t^k \vec{v}\|_{L_p(E_4^T)} \leq C(p,T)\|\vec{v}^{\,0}\|_{L_p(R^3)} \quad . \qquad (41)$$

We denote the lth order derivative with respect to x_1, x_2, x_3 by D_x^l , that is,

$$D_x^l = \frac{\partial^l}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}} .$$

The convolution (36) then possess the property

$$D_x^l \left(\vec{v}^{\,0} * \Gamma\right)_{R^3} = \left(D_x^l \vec{v}^{\,0} * \Gamma\right)_{R^3} \quad . \qquad (42)$$

Thus, on account of (41), (42) we have the following expression for the derivatives of $\vec{v}(x,t)$:

$$D_t^k D_x^l \vec{v}(x,t) = D_x^l \vec{v}^{\,0}(x) D_t^k \Phi(t) + \left(D_x^l \vec{v}^{\,0} * D_t^k \Gamma\right)_{R^3} . \quad (43)$$

Bearing in mind (43) and the property of the kernel (29), we have the following estimate for the derivatives of $\vec{v}(x,t)$:

$$\|D_t^k D_x^l \vec{v}\|_{L_p(E_4^T)} \leq C(p,T)\|D_x^l \vec{v}^{\,0}\|_{L_p(R^3)} . \qquad (44)$$

It remains to find an estimate for $P(x,t)$. We calculate ∇P for the system (19). We cannot differentiate (34) directly with respect to x_i , since the kernels in the integrand would then have a locally non-integrable singularity. In (34) we perform the substitution $y - x = x$, $dy = dx$:

$$P(x,t) = \frac{1}{4p}\iiint v_3^0(x+x)\frac{\partial K}{\partial x_3}dx \ , \quad K(x,t) = -\frac{1}{r}\int_0^t J_0(t-t)J_0\left(\frac{r}{r}t\right)dt$$

(henceforth we shall sometimes omit the argument $\left(r/r\right)t$ in Bessel functions).

As in calculation of $\vec{v}(x,t)$, we remove the singular point in the space R^3 by the ball of radius e and with boundary S_e . We denote the rest of the domain by Ω_e . We integrate by parts, calculating each component of ∇p separately.

We have

$$\frac{\partial P}{\partial x_1} = \lim_{e \to 0} \frac{1}{4p} \iiint_{\Omega_e} \frac{\partial v_3^0}{\partial x_1} \frac{\partial K}{\partial x_3} dx =$$

$$= -\lim_{e \to 0} \frac{1}{4p} \iiint_{\Omega_e} v_3^0(y) \frac{\partial^2 K}{\partial y_1 \partial y_3} dy - \lim_{e \to 0} \frac{1}{4p} \iint_{S_e} v_3^0(y) \frac{\partial K}{\partial y_3} \cos(n, y_1) ds =$$

$$= -\lim_{e \to 0} \frac{1}{4p} \iiint_{\Omega_e} v_3^0(y) \frac{\partial^2 K}{\partial y_1 \partial y_3} dy -$$

$$-\frac{v_3^0(x)}{4p} \lim_{e \to 0} \iint_{S_e} \int_0^t \left[\frac{(y_3 - x_3)(y_1 - x_1)}{r^4} J_0 - \frac{(r^2 - r^2)(y_1 - x_1)}{r^5} J_0' \right] J_0(t - t) dt ds$$

The surface integral is zero since the integrand is odd. Thus we have

$$\frac{\partial P}{\partial x_1} = -\lim_{e \to 0} \frac{1}{4p} \iiint_{\Omega_e} v_3^0(y) \frac{\partial^2 K}{\partial y_1 \partial y_3} dy \quad ,$$

and, analogously,

$$\frac{\partial P}{\partial x_2} = -\lim_{e \to 0} \frac{1}{4p} \iiint_{\Omega_e} v_3^0(y) \frac{\partial^2 K}{\partial y_2 \partial y_3} dy \quad .$$

For the third component of grad p we have

$$\frac{\partial P}{\partial x_3} = -\lim_{e \to 0} \frac{1}{4p} \iiint_{\Omega_e} v_3^0(y) \frac{\partial^2 K}{\partial y_3^2} dy -$$

$$-\frac{v_3^0(x)}{4p} \lim_{e \to 0} \int_0^t J_0(t - t) \iint_{S_e} \left[\frac{(y_3 - x_3)^2}{r^4} J_0 - \frac{(r^2 - r^2)(y_3 - x_3)t}{r^5} J_0' \right] ds dt \quad . \qquad (45)$$

The last line in (45) may be written as

$$-v_3^0(x) M(t) \text{ , where}$$

$$M(t) = \int_0^t J_0(t - t) \left[\int_0^1 (4z^2 - 1) J_0(tz) dz \right] dt \quad .$$

Thus it is possible to write $\frac{\partial P}{\partial x_i}$ as the corresponding convolutions :

$$\frac{\partial P}{\partial x_1} = (v_3^0 * K_{13})_{R^3}$$

$$\frac{\partial P}{\partial x_2} = (v_3^0 * K_{23})_{R^3} \qquad (46)$$

$$\frac{\partial P}{\partial x_3} = (v_3^0 * K_{33})_{R^3} - v_3^0(x) M(t) \quad ,$$

where

$$K_{i3} = \int_0^t J_0(t - t) \tilde{\Omega}_{i3}(x, t) dt \quad ,$$

$$\tilde{\Omega}_{13}(x, t) = \frac{1}{4p} \frac{\partial^2}{\partial x_1 \partial x_3} \left(-\frac{1}{r} J_0\left(\frac{r}{r} t \right) \right) =$$

$$= \frac{1}{4p} \left[\frac{-3x_1 x_3}{r^5} J_0 - \frac{t x_1 (3x_3^2 - 2x_1^2 - 2x_2^2)}{r^6} J_0' + \frac{x_1 x_3 t^2 (x_1^2 + x_2^2)}{r^7} J_0'' \right] ,$$

$$\tilde{\Omega}_{23}(x,t)=\frac{1}{4p}\left[\frac{-3x_2x_3}{r^5}J_0-\frac{tx_2\left(3x_3^2-2x_1^2-2x_2^2\right)}{r^6}J_0'+\frac{x_2x_3t^2\left(x_1^2+x_2^2\right)}{r^7}J_0''\right],$$

$$\tilde{\Omega}_{33}(x,t)=\frac{1}{4p}\left[\frac{\left(x_1^2+x_2^2-2x_3^2\right)}{r^5}J_0+\frac{5tx_3\left(x_1^2+x_2^2\right)}{r^6}J_0'-\frac{t^2\left(x_1^2+x_2^2\right)^2}{r^7}J_0''\right] \quad .$$

It can be proved that the kernels in (46) satisfy all the conditions 1)-3) of the Calderón-Zygmund Theorem. Conditions 1) and 2) are obvious. Let us verify condition 3), that is, that the integrals of all $\tilde{\Omega}_{i3}$ over the unit sphere are zero. For,

$$\iint_{r=1}\tilde{\Omega}_{13}ds=0 \quad , \quad \iint_{r=1}\tilde{\Omega}_{23}ds=0 \quad ,$$

since the kernels in question are odd with respect to x_1 or x_2. We prove that

$$\iint_{r=1}\tilde{\Omega}_{33}ds=0 \quad .$$

Using once again the previous change of variables, we have

$$\iint_{r=1}\tilde{\Omega}_{33}ds=\int_0^{p/2}\left[\begin{array}{l}\left(\sin^2 q-2\cos^2 q\right)J_0(t\cos q)-\\ -t^2\sin^4 q\,J_0''(t\cos q)+5t\cos q\sin^2 q\,J_0'(t\cos q)\end{array}\right]\sin q\,dq=$$

$$=\int_0^1\left(1-3z^2\right)J_0(tz)-t^2\left(1-z^2\right)^2J_0''(tz)+5tz\left(1-z^2\right)J_0'(tz)dz \,. \qquad (47)$$

Integrating by parts in the middle term in (47), we obtain

$$-\int_0^1 t^2\left(1-z^2\right)^2J_0''(tz)_0dz=-4\int_0^1 tz\left(1-z^2\right)J_0'(tz)dz \,.$$

Thus, the integral in (47) transforms into

$$\int_0^1\left(1-3z^2\right)J_0(tz)-tz\left(1-z^2\right)J_0'(tz)dz \,.$$

Integrating by parts, we have

$$\int_0^1\left(1-3z^2\right)J_0(tz)dz=\int_0^1 tz\left(1-z^2\right)J_0'(tz)dz \,,$$

and, finally, the condition $\iint_{r=1}\tilde{\Omega}_{33}ds=0$ is verified. Thus, on the basis of the Calderón-Zygmund Theorem, in the layer $E_4^T=\{-\infty<x_i<+\infty,\ 0\le t\le T\}$ (and also on each cross-section t =const) we have the following estimate :

$$\|\nabla P\|_{L_p\left(E_4^T\right)}\le C(p,T)\left\|\vec{v}^{\,0}\right\|_{L_p\left(R^3\right)} \qquad (48)$$

for $1<p<\infty$.

It is easy to see that the convolutions in (46) have properties analogous to (42), (43). As a result we have

$$\left\| D_t^k D_x^l \nabla P \right\|_{L_p\left(E_4^T\right)} \le C(p,T) \left\| D_x^l \vec{v}^{\,0} \right\|_{L_p\left(R^3\right)} . \quad (49)$$

If we denote by $W_{p,t,x}^{k,l}\left(E_4^T\right)$ the Sobolev space of functions having k derivatives with respect to t and l derivatives with respect to x which are pth power summable, then we have proved the following theorem.

Theorem 6.

If the initial data satisfy $\vec{v}^{\,0}(x) \in W_p^l\left(R^3\right)$ and if $\vec{v}(x,t)$, $P(x,t)$ is a solution of the problem (19),(23),(24) for which the norms given below are finite, then the following estimates will hold :

$$\|\vec{v}\|_{W_{p,t,x}^{k,l}\left(E_4^T\right)} \le C_1(p,T) \left\|\vec{v}^{\,0}\right\|_{W_p^l\left(R^3\right)} , \quad (50)$$

$$\|\nabla P\|_{W_{p,t,x}^{k,l}\left(E_4^T\right)} \le C_2(p,T) \left\|\vec{v}^{\,0}\right\|_{W_p^l\left(R^3\right)} , \quad (51)$$

where the constants $C_i(p,T)$ depend only on p and, in general, on T (where $0 \le t \le T < \infty$) , k and l .

6. Conclusions

The importance of construction of a Weyl sequence for such problems is that, due to the last condition of (11), a Weyl sequence is “almost” a solution of a system of partial differential equations. And, for ? belonging to the essential spectrum of B , the Weyl sequence represent explicit examples of non-uniqueness of the solutions, due to the arbitrarity of the function ? .

As we have seen, the solutions of the considered problems are closely related to the function

$$V = \frac{1}{r} J_0\left(\frac{r}{r} t\right) = \frac{1}{r} J_0(t \cos q) .$$

Let us discuss the conduct of the function V as a function of t . We consider a sphere of a constant radius. On the sphere, for every t , the function V depends only on the polar angle ? . The argument of the Bessel function on the sphere changes from 0 to t . With t growing, we will have more and more waves generated by maxima and minima of the Bessel function, all of them situated between the pole and the equator of the sphere. The waves will appear on the pole and then will move towards the equator, accumulating but not disappearing. Thus large waves will generate more and more short ones.

The remarkable analogy of gravitational and rotational waves discussed above could serve as an example of how mathematical description of physical forces of different origin may help us to understand the unity of the Nature’s manifestations.

7. References

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