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Numerical Strategies for the System of Second Order IVPs Using the RK-Butcher Algorithms

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Abstract

In this paper, a new method of analysis for second order initial value problems using the RK-Butcher algorithms is presented. To illustrate the effectiveness of the RK-Butcher algorithms, seven problems of different kinds have been considered and the solutions were obtained using the RK method based on Arithmetic mean (RKAM), Centroidal mean (RKCeM) and the RK-Butcher Algorithms and are compared with the exact solutions of the seven problems. Stability regions for the RK-Butcher algorithm, RKAM and RKCeM methods are presented. Error graphs for the second order initial value problems have been presented in a graphical form to show the efficiency of this RK-Butcher algorithm. This RK-Butcher algorithm can be easily implemented in a digital computer and the solution can be obtained for any length of time.

Keywords: Runge-Kutta method: Arithmetic mean; Centroidal mean; RK-Butcher Algorithms and System of second order IVPs.

1 Introduction

Runge –Kutta (RK) methods are being applied to determine numerical solutions for the problems, which are modeled as Initial Value Problems (IVP's) involving differential equations that arise in the fields of Science and Engineering by Alexander and Coyle [1], Dekker and Verwer [6] Evans [7], Evans and Yaakub [8,9], Hairer and Wanner [12], Murugesan et al [14-18], and Yaakub and Evans [23,24]. Though the RK method had been introduced at the turn of the 20th century, research in this area is still very active and its applications are enormous. This is because of its nature of extending accuracy in the determination of approximate solutions and its flexibility.

Runge-Kutta methods have become very popular, both as computational techniques as well as subject for research, which were discussed by Butcher [4,5], Lambert [13] and Shampine [22]. This method was derived by Runge about the year 1894 and extended by Kutta a few years later. They developed algorithms to solve differential equations efficiently and yet are the equivalent of approximating the exact solutions by matching 'n' terms of the Taylor series expansion

Runge –Kutta (RK) algorithms have always been considered superb tools for the numerical integration of Ordinary Differential Equations (ODE's). The fact that RK methods are self-starting, easy to program, and show extreme accuracy and versatility in ODE problems has led to their continuous analysis and use in mathematical research. One of the most exciting developments in RK usage has been the discovery that by judicious re-arrangement of interim values of the RK predictors one can obtain a second predictor of one order less. These two equations are generally referred to as an RK pair. Fehlberg [10] was among the first to suggest on theoretical grounds that the

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difference between the two predictors would be directly proportional to the local truncation error (LTE). The unusual success of the Fehlberg approach was addressed in the popular text by Forsythe et al [11] and cited as the "state of the art" of RK code. The LTE is then used as a test to see whether a step has been successful, and if not, the step size is reduced (usually halved) until the LTE passes the tolerance requirement. The beauty of the RK pair is that it requires no extra function evaluations, which is the most time consuming aspect of all ODE solvers. This breakthrough initiated a search for RK algorithms of higher and higher order for better error estimates.

Butcher [4,5] derived the best RK pair a long with an error estimate and by all statistical measures it appeared as the RK-Butcher algorithms. This RK-Butcher algorithm is nominally considered sixth order since it requires six function evaluations (it looks like it is sixth order method, but it is a fifth order method only), but in actual practice the "working order" is closer to five (fifth order) and the accuracy of results while solving the problems exceeds all the other algorithms examined including RK-Fehlberg, RK-Centroidal Mean (RKCeM) and RK-Arithmetic Mean (RKAM).

Morris Bader [2, 3] introduced the RK-Butcher algorithms for finding the truncation error estimates and intrinsic accuracies and the early detection of stiffness in coupled differential equations that arises in theoretical chemistry problems. Most recently, Murugesan et al [19] and Park et al [20] applied the RK-Butcher algorithm for an industrial robot arm control problem and optimal control of linear singular systems. In this paper, we introduce the RK-Butcher algorithm for finding the numerical solution of the second order IVPs with more accuracy.

2 RK-Butcher Algorithms

The normal order of an RK algorithm is the approximate number of leading terms of an infinite Taylor series, which calculates the trajectory of a moving point, which was discussed by Shampine and Gordon [21]. The remainder of the infinite sum excluded is referred to as the local truncation error (LTE). RK algorithms are forward looking predictors, that is, they use no information from preceding steps to predict the future position of a point. For this reason, they require a minimum of input data and consequently are very easy to program and simple to use.

The general p-stage Runge-Kutta method for solving an IVP

$$y' = f(x, y) \tag{1}$$

with the initial condition $y(x_0) = y_0$ is defined by $y_{n+1} = y_n + h\sum_{i=1}^{p} b_i k_i$

where $k_i = f\left(x_n + c_i h, y_n + h\sum_{j=1}^{p} a_{ij} k_j \right)$ and $c_i = \sum_{j=1}^{p} a_{ij}; i = 1,2,....p$ with c and b are p dimensional vectors and $A(a_{ij})$ be the p×p matrix. Then the Butcher array is of the form

c_1	a_{11}			
c_2	a_{21}	a_{22}		
c_3	a_{31}	a_{32}	a_{33}	
.	.	.	.	
.	.	.	.	
.	.	.	.	
..				
.	.	.	.	.
c_p	a_{p1}	a_{p2}..........	$a_{p,p-1}$	.$a_{p,p}$
	b_1	b_2	.b_{p-1}	b_p

Table – 1. Butcher array table

Let h denote the interval between equidistant values of x, then the RK-Butcher algorithm of the $(n+1)^{th}$ increment in y is computed as

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf\left(x_n + \frac{h}{4}, y_n + \frac{k_1}{4}\right)$$
$$k_3 = hf\left(x_n + \frac{h}{4}, y_n + \frac{k_1}{8} + \frac{k_2}{8}\right)$$
$$k_4 = hf\left(x_n + \frac{h}{2}, y_n - \frac{k_2}{2} + k_3\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$
$$k_5 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{3k_1}{16} + \frac{9k_4}{16}\right)$$
$$k_6 = hf\left(x_n + h, y_n - \frac{3k_1}{7} + \frac{2k_2}{7} + \frac{12k_3}{7} - \frac{12k_4}{7} + \frac{8k_5}{7}\right)$$

5th order predictor $y_{n+1} = y_n + \frac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)$

4th order predictor $y^*_{n+1} = y_n + \frac{h}{6}(k_1 + 4k_4 + k_6)$

Local Truncation Error Estimate (EE) $EE = y_{n+1} - y^*_{n+1}$

Then the $(n+1)^{th}$ increment in y and $\dot{y}$ are computed as follows:

$$y_{n+1} = y_n + \Delta y$$
$$\dot{y}_{n+1} = \dot{y} + \Delta\dot{y}$$

$$k_1 = \dot{y}_n$$
$$k_2 = \dot{y}_n + \frac{hl_1}{4}$$
$$k_3 = \dot{y}_n + \frac{hl_1}{8} + \frac{hl_2}{8}$$
$$k_4 = \dot{y}_n - \frac{hl_2}{2} + hl_3$$
$$k_5 = \dot{y}_n + \frac{3hl_1}{16} + \frac{9hl_4}{16}$$
$$k_6 = \dot{y}_n - \frac{3hl_1}{7} + \frac{2hl_2}{7} + \frac{12hl_3}{7} - \frac{12hl_4}{7} + \frac{8hl_5}{7}$$

where $\Delta y = \frac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)$

and $$\Delta\dot{y} = \frac{h}{90}\left(7l_1 + 32l_3 + 12l_4 + 32l_5 + 7l_6\right)$$

$$l_1 = f\left(x_n, y_n, \dot{y}_n\right)$$
$$l_2 = f\left(x_n + \frac{h}{4}, y_n + \frac{hk_1}{4}, \dot{y}_n + \frac{hl_1}{4}\right)$$
$$l_3 = f\left(x_n + \frac{h}{4}, y_n + \frac{hk_1}{8} + \frac{hk_2}{8}, \dot{y}_n + \frac{hl_1}{8} + \frac{hl_2}{8}\right)$$
$$l_4 = f\left(x_n + \frac{h}{2}, y_n - \frac{hk_2}{2} + hk_3, \dot{y}_n - \frac{hl_2}{2} + hl_3\right)$$
$$l_5 = f\left(x_n + \frac{3h}{4}, y_n + \frac{3hk_1}{16} + \frac{9hk_4}{16}, \dot{y}_n + \frac{3hl_1}{16} + \frac{9hl_4}{16}\right)$$
$$l_6 = f\left(x_n + h, y_n - \frac{3hk_1}{7} + \frac{2hk_2}{7} + \frac{12hk_3}{7} - \frac{12hk_4}{7} + \frac{8hk_5}{7}, \dot{y}_n - \frac{3hl_1}{7} + \frac{2hl_2}{7} + \frac{12hl_3}{7} - \frac{12hl_4}{7} + \frac{8hl_5}{7}\right)$$

System of second order linear differential equations originates in the form of mathematical formulation of problems in mechanics, electronic circuits and electrical networks, etc. Hence, the concept of solving a second order equation using the RK-Butcher algorithm can also be extended to find the numerical solution of the system of second order equations as given below.

Consider the second order IVPs

$$\ddot{y}_i = f_i\left(x, y_i, \dot{y}_i\right), i = 1,2,......m$$

with

$$y_j\left(x_0\right) = y_{j0} \qquad \text{for all } j = 1,2,....m$$
$$\dot{y}_j\left(x_0\right) = \dot{y}_{j0}$$

Then the RK-Butcher algorithm to determine $\dot{y}_j$, $j = 1,2,...,m$ are given by

$$y_{jn+1} = y_{jn} + \frac{h}{90}\left(7k_{1j} + 32k_{3j} + 12k_{4j} + 32k_{5j} + 7k_{6j}\right)$$

and

$$\dot{y}_{jn+1} = \dot{y}_{jn} + \frac{h}{90}\left(7l_{1j} + 32l_{3j} + 12l_{4j} + 32l_{5j} + 7l_{6j}\right)$$

where

$$k_{1j} = \dot{y}_{jn}$$
$$k_{2j} = \dot{y}_{jn} + \frac{hl_{1j}}{4}$$
$$k_{3j} = \dot{y}_{jn} + \frac{hl_{1j}}{8} + \frac{hl_{2j}}{8}$$
$$k_{4j} = \dot{y}_{jn} - \frac{hl_{2j}}{2} + hl_{3j}$$
$$k_{5j} = \dot{y}_{jn} + \frac{3hl_{1j}}{16} + \frac{9hl_{4j}}{16}$$
$$k_{6j} = \dot{y}_{jn} - \frac{3hl_{1j}}{7} + \frac{2hl_{2j}}{7} + \frac{12hl_{3j}}{7} - \frac{12hl_{4j}}{7} + \frac{8hl_{5j}}{7}$$

$$l_{1j} = f\left(x_n, y_{jn}, \dot{y}_{jn}\right)$$
$$l_{2j} = f\left(x_n + \frac{h}{4}, y_{1n} + \frac{hk_{11}}{4}, y_{2n} + \frac{hk_{12}}{4}, \ldots, y_{mn} + \frac{hk_{1m}}{4}, \dot{y}_{1n} + \frac{hl_{11}}{4}, \dot{y}_{2n} + \frac{hl_{12}}{4}, \ldots, \dot{y}_{mn} + \frac{hl_{1m}}{4}\right)$$
$$l_{3j} = f\left(\begin{array}{l} x_n + \frac{h}{4}, y_{1n} + \frac{hk_{11}}{8} + \frac{hk_{21}}{8}, y_{2n} + \frac{hk_{12}}{8} + \frac{hk_{22}}{8}, \ldots, y_{mn} + \frac{hk_{1m}}{8} + \frac{hk_{2m}}{8}, \dot{y}_{1n} + \frac{hl_{11}}{8} + \frac{hl_{21}}{8}, \\ \dot{y}_{2n} + \frac{hl_{12}}{8} + \frac{hl_{22}}{8}, \ldots, \dot{y}_{mn} + \frac{hl_{1m}}{8} + \frac{hl_{2m}}{8} \end{array}\right)$$

$$l_{4j} = f\left(\begin{array}{l} x_n + \frac{h}{2}, y_{1n} - \frac{hk_{21}}{2} + hk_{31}, y_{2n} - \frac{hk_{22}}{2} + hk_{32}, \ldots, y_{mn} - \frac{hk_{2m}}{2} + hk_{3m}, \dot{y}_{1n} - \frac{hl_{21}}{2} + hl_{31}, \\ \dot{y}_{2n} - \frac{hl_{22}}{2} + hl_{32}, \ldots, \dot{y}_{mn} - \frac{hl_{m1}}{2} + hl_{m1} \end{array}\right)$$
$$l_{5j} = f\left(\begin{array}{l} x_n + \frac{3h}{4}, y_{1n} + \frac{3hk_{11}}{16} + \frac{9hk_{41}}{16}, y_{2n} + \frac{3hk_{12}}{16} + \frac{9hk_{42}}{16}, \ldots, y_{mn} + \frac{3hk_{1m}}{16} + \frac{9hk_{4m}}{16}, \dot{y}_{1n} + \frac{3hl_{11}}{16} + \frac{9hl_{41}}{16}, \\ \dot{y}_{2n} + \frac{3hl_{12}}{16} + \frac{9hl_{42}}{16}, \ldots, \dot{y}_{mn} + \frac{3hl_{1m}}{16} + \frac{9hl_{4m}}{16} \end{array}\right)$$
$$l_{6j} = f\left(\begin{array}{l} x_n + h, y_{1n} - \frac{3hk_{11}}{7} + \frac{2hk_{21}}{7} + \frac{12hk_{31}}{7} - \frac{12hk_{41}}{7} + \frac{8hk_{51}}{7}, y_{2n} - \frac{3hk_{12}}{7} + \frac{2hk_{22}}{7} + \frac{12hk_{32}}{7} \\ - \frac{12hk_{42}}{7} + \frac{8hk_{52}}{7}, \ldots, y_{mn} - \frac{3hk_{1m}}{7} + \frac{2hk_{2m}}{7} + \frac{12hk_{3m}}{7} - \frac{12hk_{4m}}{7} + \frac{8hk_{5m}}{7}, \\ \dot{y}_{1n} - \frac{3hl_{11}}{7} + \frac{2hl_{21}}{7} + \frac{12hl_{31}}{7} - \frac{12hl_{41}}{7} + \frac{8hl_{51}}{7}, \dot{y}_{12} - \frac{3hl_{12}}{7} + \frac{2hl_{22}}{7} + \frac{12hl_{32}}{7} - \frac{12hl_{42}}{7} + \frac{8hl_{52}}{7}, \ldots, \\ \dot{y}_{mn} - \frac{3hl_{1m}}{7} + \frac{2hl_{2m}}{7} + \frac{12hl_{3m}}{7} - \frac{12hl_{4m}}{7} + \frac{8hl_{5m}}{7} \end{array}\right)$$
for all $j = 1, 2, \ldots, m.$

Then the formation of the Butcher array of the above equation (2) takes the following form

0						
$\frac{1}{4}$	$\frac{1}{4}$					
$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$				
$\frac{1}{2}$	0	$-\frac{1}{2}$	1			
$\frac{3}{4}$	$\frac{3}{16}$	0	0	$\frac{9}{16}$		
1	$-\frac{3}{7}$	$\frac{2}{7}$	$\frac{12}{7}$	$-\frac{12}{7}$	$\frac{8}{7}$	
	$\frac{7}{90}$	0	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$
	$\frac{1}{6}$	0	0	$\frac{4}{6}$	0	$\frac{1}{6}$

Table –2. Butcher array for equation (2)

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

Table –3. Butcher array for Runge-Kutta Arithmetic Mean (RKAM)

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & \frac{1}{24} & \frac{11}{24} & & \\
1 & \frac{11}{132} & -\frac{25}{132} & -\frac{73}{66} & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

Table –4. Butcher array for Runge-Kutta Centroidal Mean (RKCeM)

This Butcher array plays a vital role in the study of stability regions and is presented in the next sections.

3 Stability Regions

Consider a test equation $\dot{y} = l y$ where l is a constant and also complex in nature and it is used to determine the stability region of the methods.

3.1 Stability Region of RK-Butcher Algorithms

$$k_1 = f(y_n) = l y_n$$

$$k_2 = f\left(y_n + \frac{hk_1}{4}\right) = l y_n\left(1 + \frac{hl}{4}\right)$$

$$k_3 = f\left(y_n + \frac{hk_1}{8} + \frac{hk_2}{8}\right) = l y_n\left(1 + \frac{hl}{8} + \frac{hl}{8}\left(1 + \frac{hl}{4}\right)\right)$$

$$k_4 = f\left(y_n - \frac{hk_2}{2} + hk_3\right) = l y_n\left(1 - \frac{hl}{2}\left(1 + \frac{hl}{4}\right) + hl\left(1 + \frac{hl}{8} + \frac{hl}{8}\left(1 + \frac{hl}{4}\right)\right)\right)$$

$$k_5 = f\left(y_n + \frac{3hk_1}{16} + \frac{9hk_4}{16}\right) = ly_n\left(1 + \frac{3hl}{16} + \frac{9hl}{16}\left(\begin{array}{l} 1 - \frac{hl}{2}\left(1 + \frac{hl}{4}\right) \\ + hl\left(1 + \frac{hl}{8} + \frac{hl}{8}\left(1 + \frac{hl}{4}\right)\right)\end{array}\right)\right)$$

$$k_6 = f\left(y_n - \frac{3hk_1}{7} + \frac{2hk_2}{7} + \frac{12hk_3}{7} - \frac{12hk_4}{7} + \frac{8hk_5}{7}\right)$$

$$= ly_n\left(\begin{array}{l} 1 - \frac{3hl}{7} + \frac{2hl}{7}\left(1 + \frac{hl}{4}\right) + \frac{12hl}{7}\left(1 + \frac{hl}{8} + \frac{hl}{8}\left(1 + \frac{hl}{4}\right)\right) \\ - \frac{12hl}{7}\left(1 - \frac{hl}{2}\left(1 + \frac{hl}{4}\right) + hl\left(1 + \frac{hl}{8} + \frac{hl}{8}\left(1 + \frac{hl}{4}\right)\right)\right) \\ + \frac{8hl}{7}\left(1 + \frac{3hl}{16} + \frac{9hl}{16}\left(1 - \frac{hl}{2}\left(1 + \frac{hl}{4}\right) + hl\left(1 + \frac{hl}{8} + \frac{hl}{8}\left(1 + \frac{hl}{4}\right)\right)\right)\right)\end{array}\right)$$

Substituting $z = hl$ we get

$$k_1 = f(y_n) = ly_n$$

$$k_2 = ly_n\left(1 + \frac{z}{4}\right)$$

$$k_3 = ly_n\left(1 + \frac{z}{8} + \frac{z}{8}\left(1 + \frac{z}{4}\right)\right)$$

$$k_4 = ly_n\left(1 - \frac{z}{2}\left(1 + \frac{z}{4}\right) + z\left(1 + \frac{z}{8} + \frac{z}{8}\left(1 + \frac{z}{4}\right)\right)\right)$$

$$k_5 = ly_n\left(1 + \frac{3z}{16} + \frac{9z}{16}\left(1 - \frac{z}{2}\left(1 + \frac{z}{4}\right) + z\left(1 + \frac{z}{8} + \frac{z}{8}\left(1 + \frac{z}{4}\right)\right)\right)\right)$$

$$k_6 = ly_n\left(\begin{array}{l} 1 - \frac{3z}{7} + \frac{2z}{7}\left(1 + \frac{z}{4}\right) + \frac{12z}{7}\left(1 + \frac{z}{8} + \frac{z}{8}\left(1 + \frac{z}{4}\right)\right) - \frac{12z}{7}\left(1 - \frac{z}{2}\left(1 + \frac{z}{4}\right) + z\left(1 + \frac{z}{8} + \frac{z}{8}\left(1 + \frac{z}{4}\right)\right)\right) \\ + \frac{8z}{7}\left(1 + \frac{3z}{16} + \frac{9z}{16}\left(1 - \frac{z}{2}\left(1 + \frac{z}{4}\right) + z\left(1 + \frac{z}{8} + \frac{z}{8}\left(1 + \frac{z}{4}\right)\right)\right)\right)\end{array}\right)$$

Then the 5th order predictor formula is

$y_{n+1} = y_n + \frac{h}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)$, Substituting the values of k_1, k_2, k_3, k_4, k_5 and k_6

then we obtain

$$y_{n+1} = y_n + \frac{hly_n}{90}\left(90 + \frac{90}{2}z + \frac{30}{2}z^2 + \frac{30}{8}z^3 + \frac{30}{40}z^4 + \frac{30}{240}z^5\right)$$

divide both sides by y_n then the stability polynomial Q(z)= y_{n+1}/y_n is given as

$$Q(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \frac{z^6}{6!}$$

Figure-1 show that the stability region of RK-Butcher algorithms. In this stability region, the range for the real part of l is –2.780 < Re (z) < 0.0.

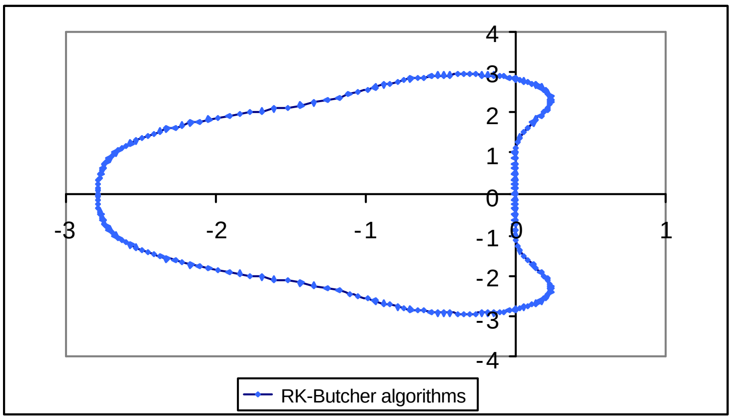

Figure- 1 Stability region for the RK-Butcher algorithms.

3.2 Stability Region of RKAM method

$$k_1 = f(y_n) = l y_n$$

$$k_2 = f\left(y_n + \frac{hk_1}{2}\right) = y_n\left(lh + \frac{1}{2}(lh)^2\right)$$

$$k_3 = f\left(y_n + \frac{hk_2}{2}\right) = y_n\left(lh + \frac{1}{2}(lh)^2 + \frac{1}{4}(lh)^3\right)$$

$$k_4 = f(y_n + hk_3) = y_n\left(lh + (lh)^2 + \frac{1}{2}(lh)^3 + \frac{1}{4}(lh)^4\right)$$

Substituting $z = hl$ we get

$$k_1 = f(y_n) = l y_n$$

$$k_2 = l y_n\left(z + \frac{1}{2}(z)^2\right)$$

$$k_3 = l y_n\left(z + \frac{1}{2}(z)^2 + \frac{1}{4}(z)^3\right)$$

$$k_4 = l y_n\left(z + (z)^2 + \frac{1}{2}(z)^3 + \frac{1}{4}(z)^4\right)$$

Then the 3rd order predictor formula is

$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$, Substituting the values of k_1, k_2, k_3 and k_4

then we obtain

$$y_{n+1} = y_n + \frac{hly_n}{6}\left(1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4\right)$$

divide both sides by y_n then the stability polynomial Q(z)= y_{n+1}/y_n is given as

$$Q(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!}$$

Figure-2 show that a stability region of RKAM method. In this stability region, the range for the real part of l is –3.463 < Re (z) < 0.0.

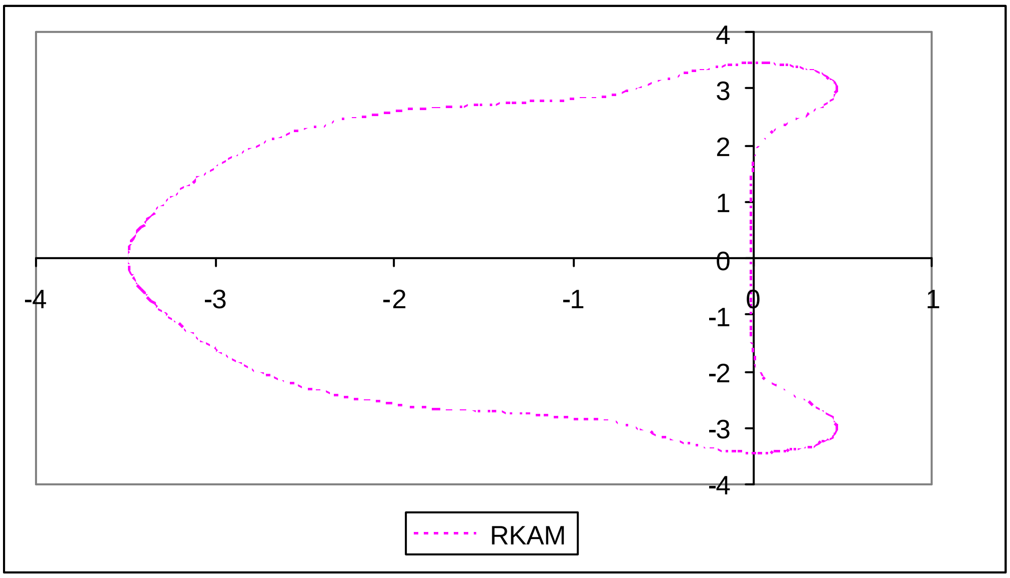

Figure- 2 Stability region for RKAM method.

3.3 Stability Region of RKCeM Method

$$k_1 = f(y_n) = ly_n$$
$$k_2 = f\left(y_n + \frac{hk_1}{2}\right) = ly_n\left(1 + \frac{hl}{2}\right)$$

$$k_3 = f\left(y_n + \frac{hk_1}{24} + \frac{11hk_2}{24}\right) = l y_n\left(1 + \frac{h\,l}{24} + \frac{11h\,l}{24}\left(1 + \frac{h\,l}{2}\right)\right)$$

$$k_4 = f\left(y_n + \frac{11hk_1}{132} - \frac{25hk_2}{132} + \frac{73hk_3}{66}\right)$$

$$= l y_n\left(1 + \frac{11h\,l}{132} - \frac{25h\,l}{132}\left(1 + \frac{h\,l}{2}\right) + \frac{73h\,l}{66}\left(1 + \frac{h\,l}{24} + \frac{11h\,l}{24}\left(1 + \frac{h\,l}{2}\right)\right)\right)$$

Substituting $z = h\,l$ we get

$$k_1 = f(y_n) = l y_n$$

$$k_2 = l y_n\left(1 + \frac{z}{2}\right)$$

$$k_3 = l y_n\left(1 + \frac{z}{2} + \frac{11z^2}{48}\right)$$

$$k_4 = l y_n\left(1 + z + \frac{11z^2}{24} + \frac{73z^3}{288}\right)$$

Then the 3rd order predictor formula is

$$y_{n+1} = y_n + \frac{2h}{9}\left(\frac{k_1^2 + k_1k_2 + k_2^2}{k_1 + k_2} + \frac{k_2^2 + k_2k_3 + k_3^2}{k_2 + k_3} + \frac{k_3^2 + k_3k_4 + k_4^3}{k_3 + k_4}\right)$$, Substituting the values of k_1, k_2, k_3 and k_4

then we obtain

$$y_{n+1} = y_n + \frac{2h\,l y_n}{18}\left(9 + \frac{18}{4}z + \frac{3}{2}z^2 + \frac{3}{8}z^3 + \frac{37}{576}z^4 + O(z^5)\right)$$

divide both sides by y_n then the stability polynomial Q(z)= y_{n+1}/y_n is given as

$$Q(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{37z^5}{5184} + O(z^6)$$

Figure-3 show that the stability region of RKCeM method. In this stability region, the range for the real part of l is -3.428 < Re (z) < 0.0.

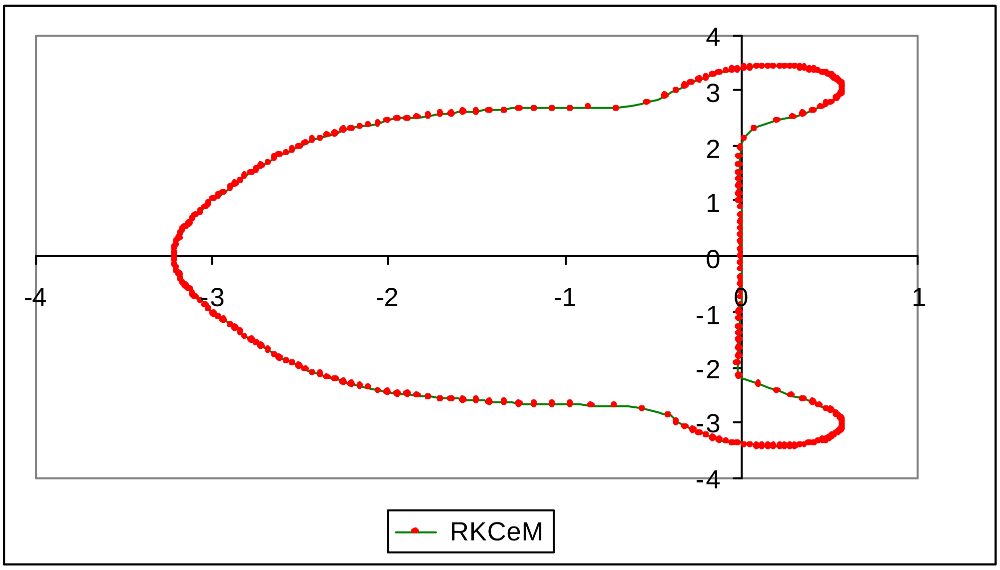

Figure- 3 Stability region for RKCeM method

Figure -4 show that a comparative study of the stability regions of the RKAM, RKCeM methods and the RK-Butcher algorithm. In this stability region, the range for the real part of l in RKAM method is –3.463 < Re (z) < 0.0, RKCeM is -3.428 < Re (z) < 0.0 and the RK-Butcher algorithm it is –2.780 < Re (z) < 0.0.

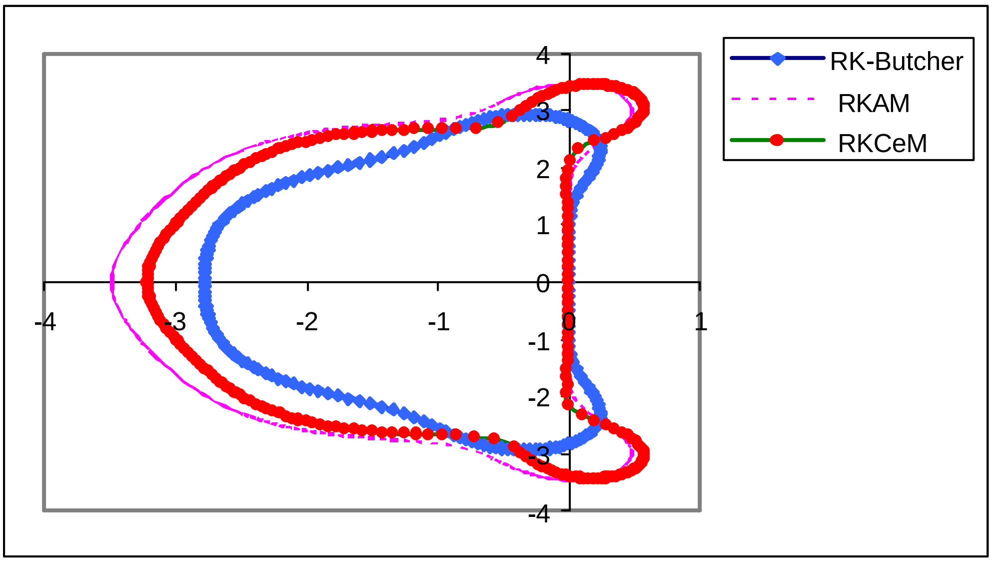

Figure- 4 Stability regions for RKAM, RKCeM and the RK-Butcher algorithms.

4 Numerical Examples

To highlight the efficiency of the RK-Butcher algorithms, we consider the following seven different problems taken from the real world applications (listed in Table -5), with step size $h = 0.2$ along with the exact solutions. The discrete solutions obtained by the three methods, RK-Butcher algorithms, RK-Centroidal mean and the RK-Arithmetic mean, the absolute errors between them are calculated and are presented in tables 6 – 12. To distinguish the effect of the errors in accordance with the exact solutions, a graphical representation is given for selected values of “x “and is presented in figures 5 -12 for all the seven problems, using three-dimensional effect.

Problems	Second order Systems	Initial Conditions	Exact Solutions
1	Linear equation $\ddot{y} = -\dot{y}$	$y(0) = 1$ $\dot{y}(0) = -1$	$y(x) = e^{-x}$
2	Linear equation $\ddot{y} = \dot{y}$	$y(0) = 1$ $\dot{y}(0) = 1$	$y(x) = e^{x}$
3	Linear equation $\ddot{y} = -\sqrt{2}\dot{y}$	$y(0) = -1/\sqrt{2}$ $\dot{y}(0) = 1$	$y(x) = -1/\sqrt{2}e^{-\sqrt{2}x}$
4	Oscillatory problem $\ddot{y} = \dot{y}\cos(x) - y\sin(x)$	$y(0) = 1$ $\dot{y}(0) = 1$	$y(x) = \exp(\sin(x))$
5	Non-linear equation $(y+1)\ddot{y} = 3(\dot{y})^2$	$y(1) = 0$ $\dot{y}(1) = -1/2$ and x > 1	$y(x) = 1/\sqrt{x} - 1$
6	Singular system $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\ddot{y} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\dot{y}$	$y_1(0) = 2$ $y_2(0) = 1$ $\dot{y}_1(0) = 0$ $\dot{y}_2(0) = 1$	$y_1(x) = 2$ $y_2(x) = -1 + 2e^{x/2}$
7	Stiff system $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\ddot{y} = \begin{bmatrix} -40 & 3 \\ 0 & 0.4 \end{bmatrix}\dot{y}$	$y_1(0) = 1$ $y_2(0) = 1$ $\dot{y}_1(0) = 0.5$ $\dot{y}_2(0) = 0.5$	$y_1(x) = 0.91875 - 0.01157e^{-40.0x}$ $+ 0.928218e^{0.4x}$ $y_2(x) = -0.25 + 1.252e^{0.4x}$

Table -5 Second order IVP's with exact solutions

Value of x	Exact Solutions	Discrete Solutions for Problem 1					
		RKAM Solutions	RKAM Error	RKCeM Solutions	RKCeM Error	RK-Butcher Solutions	RK-Butcher Error
0.00	1.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000
0.60	0.5488116145	0.5488168001	0.0000051856	0.5488117933	0.0000001788	0.5488116741	0.0000000596
1.20	0.3011941910	0.3011999130	0.0000057220	0.3011943996	0.0000002086	0.3011942506	0.0000000596
1.80	0.1652988493	0.1653035730	0.0000047237	0.1652990431	0.0000001937	0.1652989239	0.0000000745

Table- 6 Solutions and Error of the problem -1 at various values of “x”.

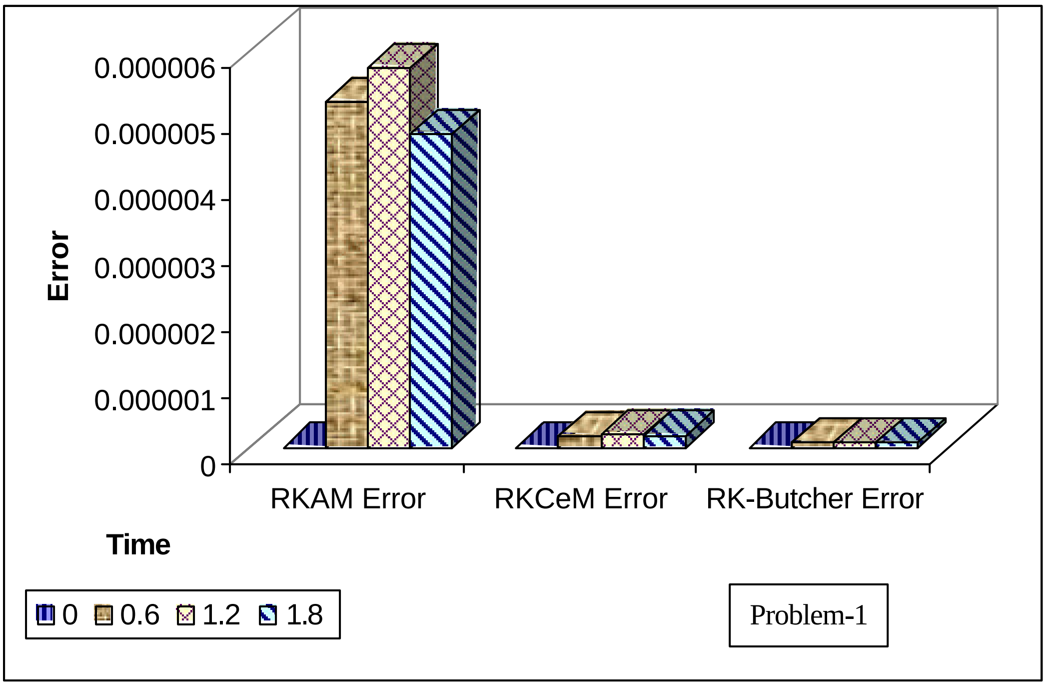

Figure- 5 Error graph for Problem -1 at various values of ‘x’

Value of x	Exact Solutions	Discrete Solutions for Problem- 2					
		RKAM Solutions	RKAM Error	RKCeM Solutions	RKCeM Error	RK-Butcher Solutions	RK-Butcher Error
0.00	1.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000
0.60	1.8221188784	1.8221064806	0.0000123978	1.8221160173	0.0000028610	1.8221188784	0.0000000000
1.20	3.3201169968	3.3200721741	0.0000448227	3.3201067448	0.0000102520	3.3201172352	0.0000002384
1.80	6.0496487617	6.0495247841	0.0001239777	6.0496191978	0.0000295639	6.0496482849	0.0000004768

Table - 7 Solutions and Error of the problem- 2 at various values of "x".

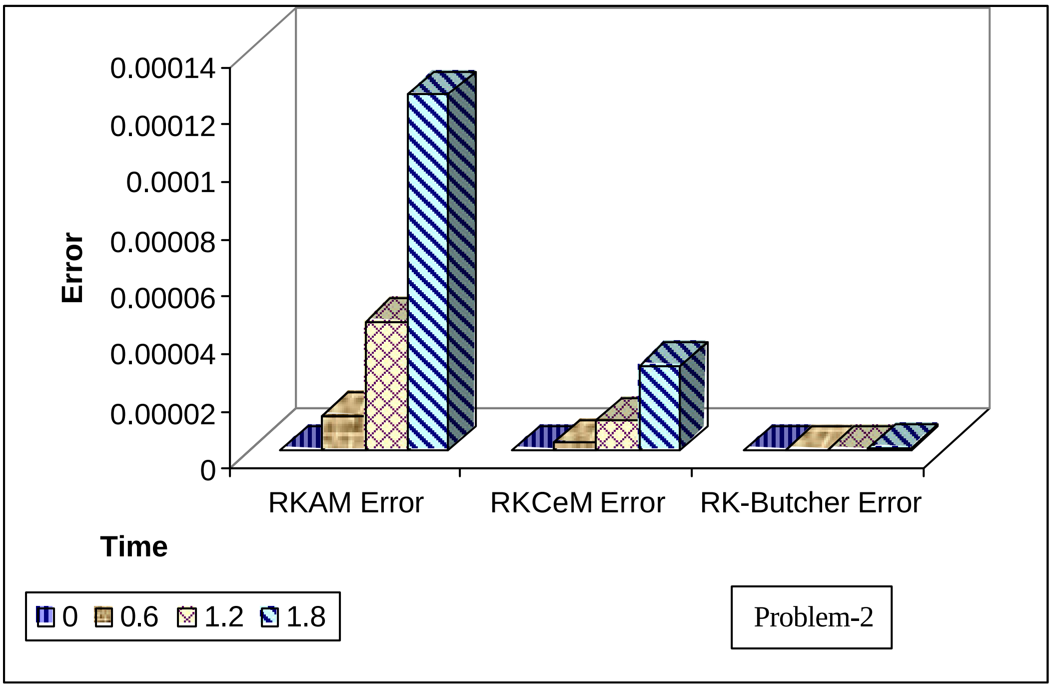

Figure- 6 Error graph for Problem -2 at various values of 'x'

Value of x	Exact Solutions	Discrete Solutions for Problem- 3					
		RKAM Solutions	RKAM Error	RKCeM Solutions	RKCeM Error	RK-Butcher Solutions	RK-Butcher Error
0.00	-0.7071067812	-0.7071067812	0.0000000000	-0.7071067812	0.0000000000	-0.7071067812	0.0000000000
0.60	-0.3026731610	-0.3026905060	0.0000173450	-0.3026728630	0.0000002980	-0.3026733100	0.0000001490
1.20	-0.1295575649	-0.1295724362	0.0000148714	-0.1295573264	0.0000002384	-0.1295577288	0.0000001639
1.80	-0.0554563925	-0.0554659516	0.0000095591	-0.0554562397	0.0000001527	-0.0554565191	0.0000001267

Table- 8 Solutions and Error of the problem- 3 at various values of “x”.

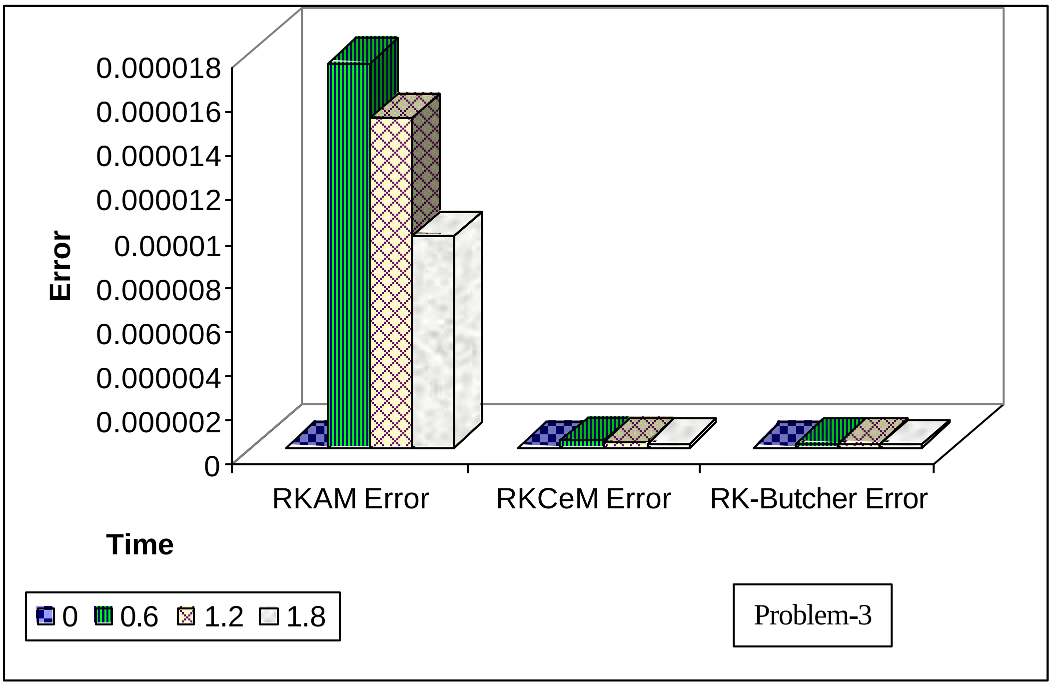

Figure -7 Error graph for Problem -3 at various values of ‘x’

Value of x	Exact Solutions	Discrete Solutions for Problem -4					
		RKAM Solutions	RKAM Error	RKCeM Solutions	RKCeM Error	RK-Butcher Solutions	RK-Butcher Error
0.00	1.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000	1.0000000000	0.0000000000
0.60	1.7588188648	1.7588198185	0.0000009537	1.7594461441	0.0006272793	1.7588191032	0.0000002384
1.20	2.5396826267	2.5396575928	0.0000250340	2.5461387634	0.0064561367	2.5396828651	0.0000002384
1.80	2.6481137276	2.6480886936	0.0000362396	2.6639122963	0.0181143284	2.6481149197	0.0000009537

Table- 9 Solutions and Error of the problem 4 at various values of “x”.

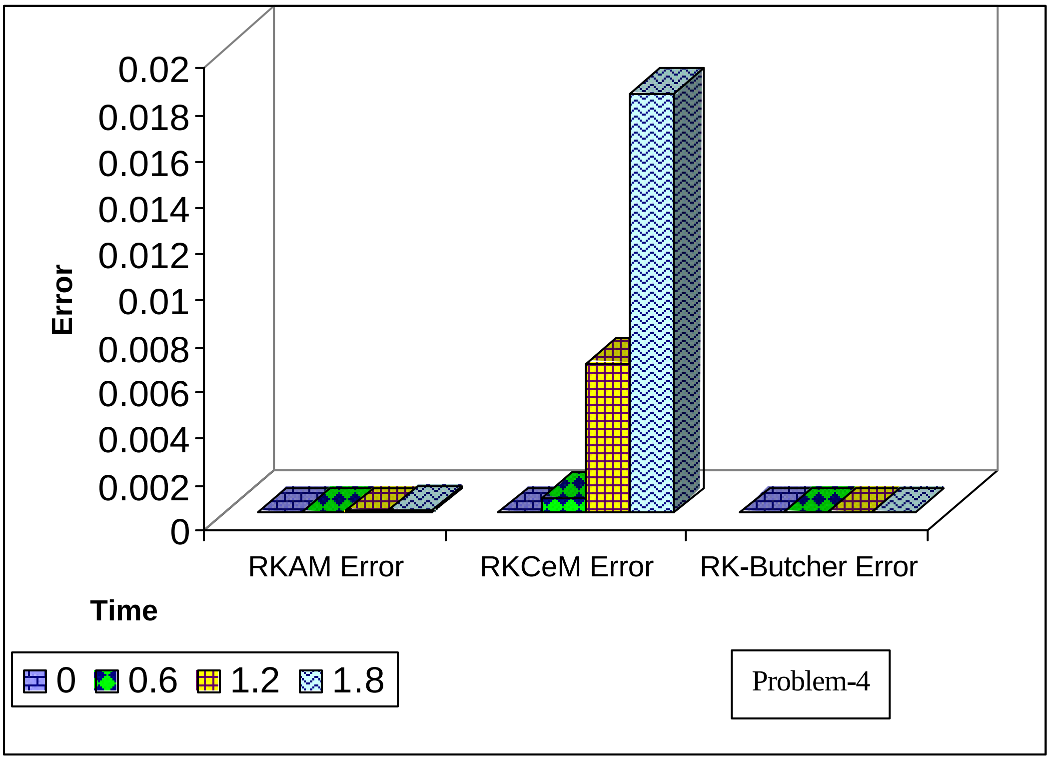

Figure - 8 Error graph for Problem - 4 at various values of ‘x’

Value of x	Exact Solutions	Discrete Solutions for Problem – 5					
		RKAM Solutions	RKAM Error	RKCeM Solutions	RKCeM Error	RK-Butcher Solutions	RK-Butcher Error
1.00	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000	0.0000000000
1.60	-0.2094306201	-0.2093935013	0.0000371188	-0.2092512697	0.0001793504	-0.2094304711	0.0000001490
2.20	-0.3258001804	-0.3257696629	0.0000305176	-0.3255960047	0.0002041757	-0.3258000910	0.0000000894
2.80	-0.4023857415	-0.4023609161	0.0000248253	-0.4021835327	0.0002022088	-0.4023856819	0.0000000596

Table - 10 Solutions and Error of the problem 5 at various values of "x".

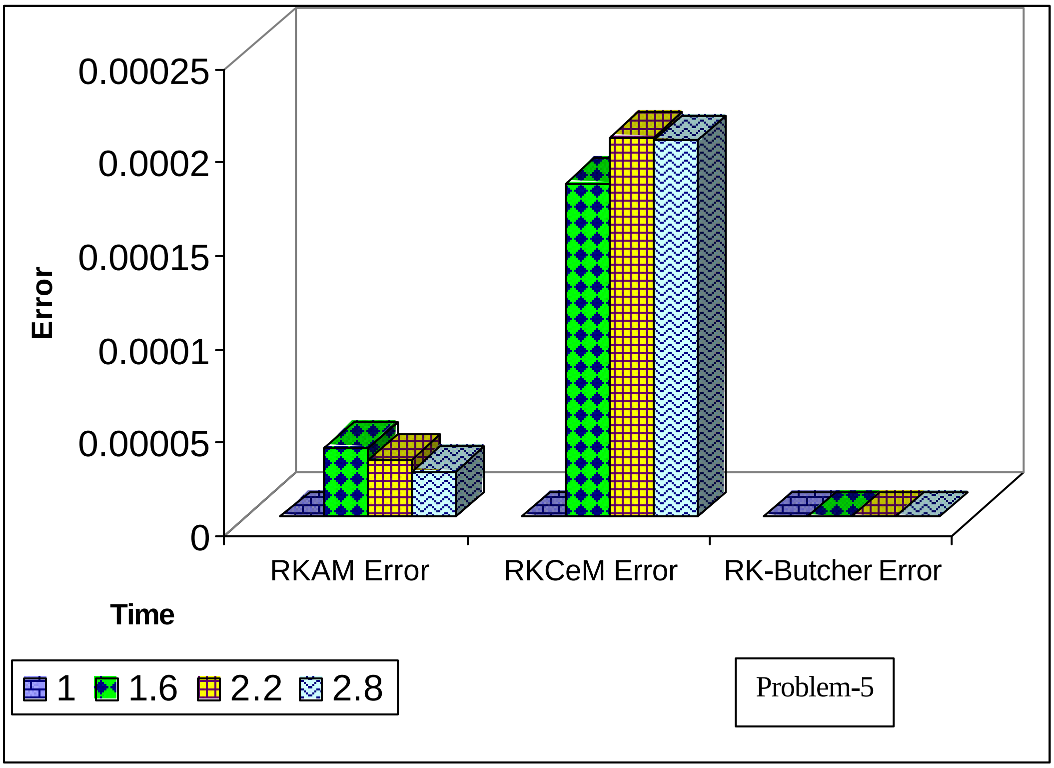

Figure - 9 Error graph for Problem- 5 at various values of 'x'

	Value of x	Exact Solutions	Discrete Solutions for Problem – 6					
			RKAM Solutions	RKAM Error	RKCeM Solutions	RKCeM Error	RK-Butcher Solutions	RK-Butcher Error
$y_1(x)$	0.00	2.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000
	0.60	2.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000
	1.20	2.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000
	1.80	2.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000	2.0000000000	0.0000000000
$y_2(x)$	0.00	1.0000000000	1.0000000002	2E-10	1.0000000005	5E-10	1.0000000000	0.0000000000
	0.60	1.6997176409	1.6997176429	2E-09	1.6997176439	3E-09	1.6997176409	0.0000000000
	1.20	2.6442377567	2.6442377601	3.4E-09	2.6442377637	7E-09	2.6442377567	0.0000000000
	1.80	3.9192063808	3.9192063888	8E-09	3.9192063928	1.2E-08	3.9192063808	0.0000000000

Table- 11 Solutions and Error of the problem-6 at various values of "x".

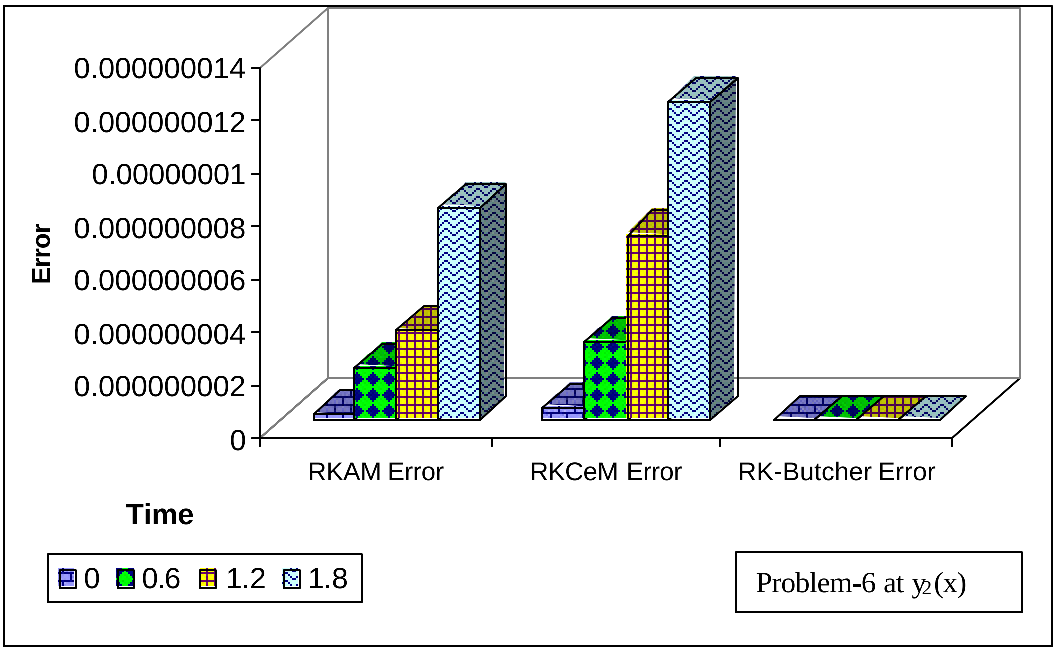

Figure -10 Error graph for Problem- 6 at various values of 'x'

	Value of x	Exact Solutions	Discrete Solutions for Problem- 7					
			RKAM Solutions	RKAM Error	RKCeM Solutions	RKCeM Error	RK-Butcher Solutions	RK-Butcher Error
$y_1(x)$	0.00	1.8353960514	1.8353960516	0	1.8353960518	0	1.8353960514	0.0000000000
	0.60	2.0987460613	2.0987460633	0	2.0987460636	0	2.0987460613	0.0000000000
	1.20	2.4188191891	2.4188191931	0	2.4188191941	0	2.4188191891	0.0000000000
	1.80	2.8257114887	2.8257114967	0	2.8257114977	0	2.8257114887	0.0000000000
				0		0		
$y_2(x)$	0.00	1.0000000000	1.0000000002	0	1.0000000004	0	1.0000000000	0.0000000000
	0.60	1.3390614986	1.3390614996	0	1.3390614998	0	1.3390614986	0.0000000000
	1.20	1.7700930834	1.7700930850	0	1.7700930853	0	1.7700930834	0.0000000000
	1.80	2.3180415630	2.3180415652	0	2.3180415658	0	2.3180415630	0.0000000000
				0		0		

Table- 12 Solutions and Error of the problem -7 at various values of "x".

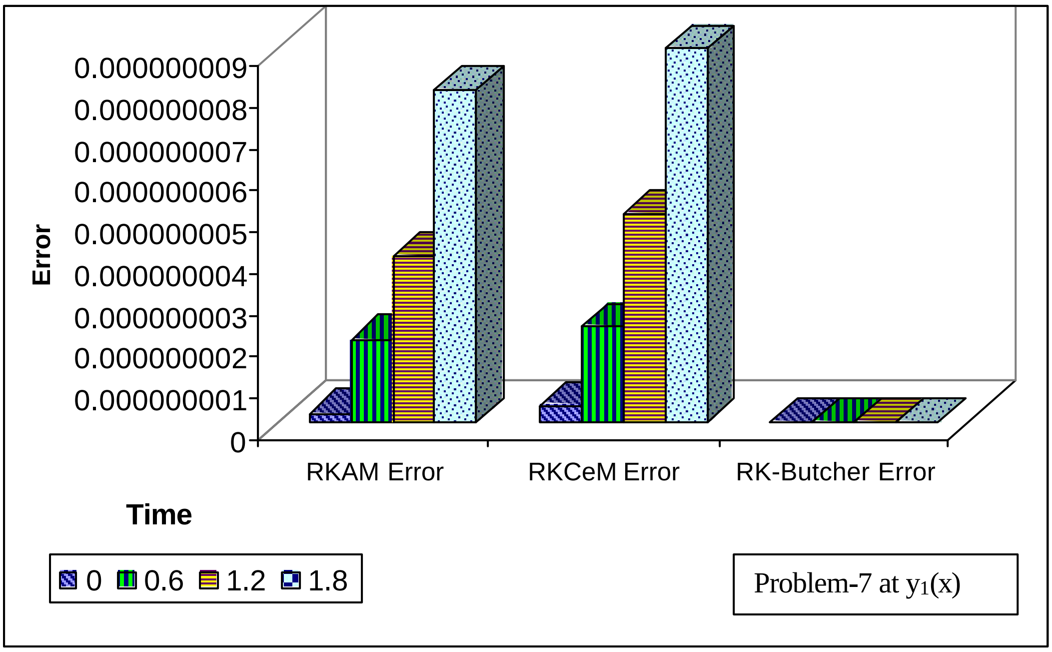

Figure -11 Error graph for Problem- 7 at various values of 'x'

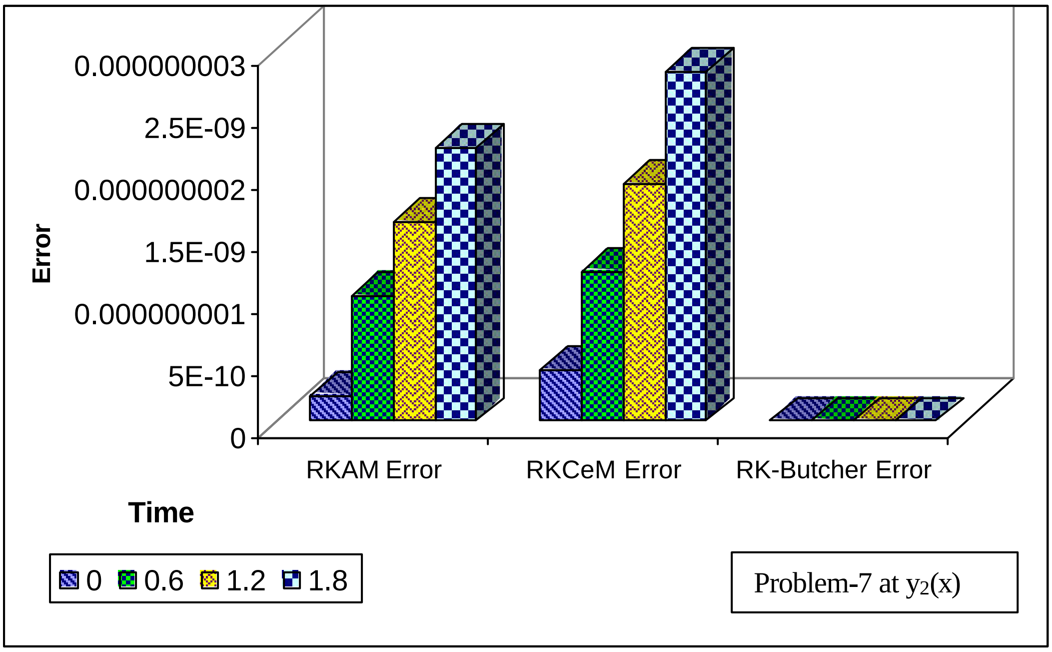

Figure- 12 Error graph for Problem -7 at various values of 'x'

5 Conclusions

The obtained discrete solution of the numerical examples for all the seven problems shows the efficiency of the RK-Butcher algorithm for solving the second order IVPs. From the tables 6 - 12, we can observe that for most of the problems, the absolute error is less (almost no error) in RK-Butcher algorithms when compared to the classical fourth order RK-method (RKAM) and the RK-Centroidal mean (RKCeM) which yields a little error, along with the exact solutions. From the figures 1 – 4, it is to be noted that the stability region of the RK-Butcher algorithm is smaller then the other RK methods and it reveals that it converges faster then the other RK methods and so it is useful for smaller time steps and the stability region of RKAM and RKCeM methods shows that it is useful for larger time steps. From the figures 5-12 and tables 6-12, one can predict that the error is very less in RK-Butcher algorithms when compared to RKAM and RKCeM methods. Hence, the RK-Butcher algorithm is more suitable for studying the system of second order IVPs.

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